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Barut–Girardello coherent states for $u(p, q)$ and $sp(N, R)$ and their macroscopic superpositions

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Abstract. The Barut–Girardello (BG) coherent states (CS) representation is extended to the noncompact algebras $u(p, q)$ and $sp(N, R)$ in (reducible) quadratic boson realizations. The $sp(N, R)$ BG CS take the form of multimode ordinary Schrödinger cat states. Macroscopic superpositions of 2^{n-1} $sp(N, R)$ CS (2^n canonical CS, $n = 1, 2, \ldots$) are pointed out which are overcomplete in the N-mode Hilbert space and the relation between the canonical CS and the $u(p, q)$ BG-type CS representations is established.

The sets of $u(p, q)$ and $sp(N, R)$ BG CS and their discrete superpositions contain many states studied in quantum optics (even and odd N-mode CS, pair CS) and provide an approach to quadrature squeezing, alternative to that of intelligent states. New subsets of weakly and strongly nonclassical states are pointed out and their statistical properties (first- and second-order squeezing, photon number distributions) are discussed. For specific values of the angle parameters and small amplitude of the canonical CS components, these states approach multimode Fock states with one, two or three bosons/photons. It is shown that eigenstates of a squared non-Hermitian operator A^2 (generalized cat states) can exhibit squeezing of the quadratures of A.

1. Introduction

Recently there has been much interest in applications and generalizations of the Barut–Girardello (BG) coherent states (CS) [1–7]. The BG CS were introduced in [8] as eigenstates of the lowering Weyl operator K_- of the algebra $su(1, 1)$. The BG CS representation has been used for explicit construction of squeezed states (SS) for the generators of the group $SU(1, 1)$ which minimize the Schrödinger uncertainty relation for two observables [1] and of eigenstates of general elements of the complexified algebra $su^C(1, 1)$ [4, 5]. The overcomplete families of eigenstates of elements of a Lie algebra were called algebraic CS [4] and algebra eigenstates [9, 5]. The idea to construct SS for quadratures of any non-Hermitian operator A as eigenstates of complex combinations $uA + vA^\dagger$ was put forward in [1], where such eigenstates $|z, u, v\rangle$ were constructed for $A = J_-$ and $A = K_-$, J_- and K_- being the Weyl lowering operators of $su(2)$ and $su(1, 1)$ Lie algebras correspondingly. The $su(1, 1)$ BG CS differ from the $SU(1, 1)$ group related CS (see [10] and references therein): the highest weight vector is the only common state, while the $su(1, 1)$ algebra related CS $|z, u, v; k\rangle$ of [1] contain the whole set of $SU(1, 1)$ group related CS with symmetry. The general set of algebra related CS always contains the corresponding group related CS with symmetry as a subset.

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Passing to other algebras it is initially important to construct the eigenstates of Weyl lowering operators, which is a direct extension of the BG definition of the $su(1,1)$ CS to the desired algebra. The aim of this paper is to construct a BG-type CS for the symplectic algebra $sp(N,R)$ and its subalgebras $u(p,q)$, $p+q=N$, in the quadratic boson representation. This $sp(N,R)$ representation is of importance in various fields of physics [11–13]. Here N is the dimension of the Cartan subalgebra, while the dimension of $sp(N,R)$ is $N(2N+1)/2$, $N=1,2,\ldots,$ [11].

We establish that the $sp(N,R)$ BG CS in quadratic boson representation takes the form of superpositions of two multimode canonical CS [10] $|\boldsymbol{\alpha}\rangle$ and $|-\boldsymbol{\alpha}\rangle$ (equation (20)). A subset of these states is found which is overcomplete in the whole Hilbert space $\mathcal{H}$ of the N-mode system. Recall that the corresponding $Sp(N,R)$ group related CS are not overcomplete in $\mathcal{H}$ since the representation is reducible. This property is a particular case of a quite general result of the overcompleteness of the eigenstates of the powers $A_j^{2^n}$ of the non-Hermitian A_j, $j=1,2,\ldots,$ provided the eigenstates $|\boldsymbol{z}\rangle$, $\boldsymbol{z}=(z_1,\ldots,z_N)$, of all A_j are overcomplete with respect to a measure independent of the phases of z_j (section 3 and appendix A.2).

Macroscopic superpositions of two canonical CS are called (ordinary) Schrödinger cat states [14, 15]. The set of the $sp(N,R)$ BG CS includes several subsets of ordinary cat states, which are extensively studied in quantum optics (see [14, 15] and references therein). We introduce *multimode squared amplitude Schrödinger cat states* as macroscopic superpositions of two $sp(N,R)$ BG CS. Unlike the ordinary cat states these superpositions, which eventually become combinations of four N-mode canonical CS, can exhibit amplitude and squared amplitude quadrature squeezing (first- and second-order quadrature squeezing or linear and quadratic squeezing) [17], and other nonclassical properties. Families of weakly and strongly nonclassical [18] cat states are pointed out as macroscopic superpositions of two $sp(N,R)$ CS. There are states in these families that tend to multimode Fock states with $0,1,2$ or 3 photons as the amplitude of their canonical CS components approaches zero. We note that, unlike the case of Robertson (Schrödinger) intelligent states (IS) [1, 6], the cat state squeezing cannot be arbitrarily strong.

Recently [7] the BG-type CS have been constructed for the $u(N-1,1)$ algebra. Here we construct overcomplete families of states for $u(p,q)$, $p+q=N$, and the related resolution unity measures as well. We show that the 'pair CS' [16] are in fact the $u(1,1)$ BG CS $|z;k\rangle$ for $k=\frac{1}{2},1\ldots,$ while the 'two-mode Schrödinger cat states' of [20] are a particular case of our $u(p,q)$ multimode squared amplitude cat states (48).

This paper is organized as follows. In section 2 a concise review of the properties of BG CS representation and its relations to the canonical one- and two-mode CS representation (or Fock–Bargman representation) [10] is given. An explicit relation between the two-mode canonical CS and the BG CS representations is obtained. Using this relation one can easily establish the coincidence between the generalized intelligent states (IS) $|z,u,v;k\rangle$ [1] and many other one- and two-mode states, constructed by other authors as eigenstates of $ua^2+va^{\dagger 2}$ or $uab+va^\dagger b^\dagger$ [21–23]. For example, for real u,v the states $|z,u,v;k=\frac{1}{2}\rangle$ coincide with the 'pair excitation–de-excitation CS' [21], while for real u,v and $k=(1+|q|)/2$ they are identical to the 'two-mode intelligent $SU(1,1)$ CS' [22].

In section 3 the BG CS are extended explicitly to the algebra $sp(N,R)$ in the (reducible) quadratic boson representation and overcomplete in whole $\mathcal{H}$ families of such states are constructed. Overcomplete families of eigenstates of the power $2n$ of Weyl operators a_ia_j are also built up. These states take the form of macroscopic superpositions with 2^n canonical CS components. The $u(p,q)$ BG-type CS are considered in section 4 (overcompleteness, resolution unity measure, particular cases and relation of their analytic representation to that of canonical CS). In section 5 the statistical properties (weak and strong nonclassicality, amplitude and squared amplitude quadrature squeezing, sub- and super-Poissonian photon statistics) of the constructed $sp(N, R)$ algebra related CS and their superpositions are discussed and illustrated by several graphics. Our analysis shows that photon number oscillations are not necessary characteristics of the nonclassicality of quantum states (neither are they sufficient [24]). We note the main difference between squeezing in intelligent SS [1, 6, 25] and in cat type SS and construct a second kind multimode squeeze operator as a map from CS $|\alpha\rangle$ to a set of cat-type multimode SS. In the appendix several statements of the main text are proved.

2. The Barut–Girardello coherent states

The property of canonical CS $|\alpha\rangle$ [10] to be eigenstates of the photon number lowering operator a, $a|\alpha\rangle = \alpha|\alpha\rangle$ (α is a complex number, $[a, a^\dagger] = 1$) was extended by BG [8] to the case of the Weyl lowering operator K_- of $su(1, 1)$ algebra. Here we briefly review some of their properties. The defining equation is

$$K_-|z; k\rangle = z|z; k\rangle \tag{1}$$

where z is the (complex) eigenvalue and k is the Bargman index. Here, and in [1], we introduced $k = -\Phi$ as a second label of the state and replaced the BG z with $z/\sqrt{2}$. For the discrete series $D^{(\pm)}(k)$ the parameter k takes the values $\pm\frac{1}{2}, \pm 1, \ldots$. The Cartan–Weyl basis operators $K_\pm = K_1 \pm \mathrm{i}K_2$, K_3 of $su(1, 1)$ obey the relations

$$[K_3, K_\pm] = \pm K_\pm \qquad [K_-, K_+] = 2K_3 \tag{2}$$

with the Casimir operator $C_2 = K_3{}^2 - (\frac{1}{2})[K_-K_+ + K_+K_-] = k(k - 1)$. The expansion of these states over the orthonormal basis of eigenstates $|k + n, k\rangle$ of K_3 ($K_3|n + k, k\rangle = (n + k)|n + k, k\rangle$, $n = 0, 1, 2, \ldots$) is

$$\begin{aligned} |z; k\rangle &= \mathcal{N}_{\mathrm{BG}}(|z|, k) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(2k + n)}} |n + k, k\rangle \equiv \mathcal{N}_{\mathrm{BG}}(|z|, k)||z; k\rangle \\ \mathcal{N}_{\mathrm{BG}}(|z|, k) &= [\Gamma(2k)/{}_0F_1(2k; |z|^2)]^{\frac{1}{2}} = \frac{|z|^{k-1/2}}{\sqrt{I_{2k-1}(2|z|)}} \end{aligned} \tag{3}$$

where ${}_0F_1(c; z)$ is the confluent hypergeometric function, $I_\nu(z)$ is the modified Bessel function of the first kind, and $\Gamma(z)$ is the gamma function [26]. The above BG states $|z; k\rangle$ are normalized to unity. Their scalar product is

$$\langle k; z|z'; k\rangle = {}_0F_1(2k; z^* z')[{}_0F_1(2k; |z|^2){}_0F_1(2k; |z'|^2)]^{-\frac{1}{2}} \tag{4}$$

and they resolve the unity (the identity operator),

$$\int \mathrm{d}\mu\,(z, k)\|z; k\rangle\langle k; z\| = 1_k \qquad \mathrm{d}\mu\,(z, k) = \frac{2}{\pi}|z|^{2k-1}K_{2k-1}(2|z|)\,\mathrm{d}^2z \tag{5}$$

where $K_\nu(x)$ is the modified Bessel function of the third kind. Note that $||z; k\rangle = \mathcal{N}_{\mathrm{BG}}^{-1}|z; k\rangle$, while in [8] these non-normalized CS were denoted as $\Gamma(2k)^{-1/2}|z\rangle$ (note also the misprint in [8]: in the formula for the measure function $\sigma(r)$ one should replace $K_{\Phi+\frac{1}{2}}(2\sqrt{2}r)$ with $K_{2\Phi+1}(2\sqrt{2}r)$ [2]). Owing to the above overcompleteness property any state $|\Psi\rangle$ can be correctly represented by the analytic function

$$F_{\mathrm{BG}}(z, k; \Psi) = \langle k, z^*|\Psi\rangle/\mathcal{N}_{\mathrm{BG}}(|z|, k) = \langle k, z^*||\Psi\rangle \tag{6}$$

which is of the growth $(1, 1)$. The orthonormalized states $|k + n, k\rangle$ are represented by monomials $z^n/\sqrt{n!\Gamma(2k + n)}$ (we note a misprint in these monomials in [4, 6]: k should be replaced by $2k$). The operators $K_\pm$ and K_3 act in the space $\mathcal{H}_k$ of analytic functions $F_{\rm BG}(z, k)$ as linear differential operators

$$K_+ = z \qquad K_- = 2k\frac{\mathrm{d}}{\mathrm{d}z} + z\frac{\mathrm{d}^2}{\mathrm{d}z^2} \qquad K_3 = k + z\frac{\mathrm{d}}{\mathrm{d}z}. \tag{7}$$

This analytic representation has been used to explicitly construct eigenstates $|z, u, v; k\rangle$ of complex combinations $uK_- + vK_+$ in paper [1].

BG have established their continuous representation for the discrete series $D^\pm(k)$, $k = \pm\frac{1}{2}, \pm 1, \ldots$. However, by inspection of their construction one can easily see that it also holds for *reducible representations* and for $\frac{1}{2} > |k| > 0$—one only has to keep in mind that the quantity 1_k in the overcompleteness relation (5) is the identity operator in the subspace $\mathcal{H}_k$, *where* $su(1, 1)$ *acts irreducibly*. The proof consists of two observations (for concreteness we take $D^+(k)$): (a) the expansions (3) are convergent and represent normalized states for $k \geqslant 0$, provided $|k, k + n\rangle$ are orthonormalized; (b) the BG measure $\mathrm{d}\mu(z, k)$ resolves the unity operators by means of $|z; k\rangle$ for $k \geqslant 0$ provided the orthonormalized set of $|k, k + n\rangle$ is complete.

It is well known that the $su(1, 1)$ algebra has one- and two-mode quadratic boson representations, which are reducible in the spaces of states of one- and two-mode systems correspondingly. The one-mode realization of $su(1, 1)$ is

$$K_- = \tfrac{1}{2}a^2, K_2 = \tfrac{1}{2}a^{\dagger 2}, K_3 = \tfrac{1}{4}(2a^\dagger a + 1). \tag{8}$$

Its quadratic Casimir operator C_2 equals $-\frac{3}{16}$, $C_2 = K_3^2 - K_1^2 - K_2^2 = k(k-1)$, the Bargman index being $k = \frac{1}{4}, \frac{3}{4}$. The two-mode representation

$$K_- = a_1 a_2 \qquad K_+ = a_1^\dagger a_2^\dagger \qquad K_3 = \tfrac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \tag{9}$$

is highly reducible (completely reducible), its irreducible components being just the representations from the discrete series $D^+(k)$, $k = \frac{1}{2}, 1, \ldots$. The whole space $\mathcal{H}$ of the two-mode system states is a direct sum of the irreducible modules $\mathcal{H}_k$. In these realizations the operators $uK_- + vK_+$, which were diagonalized in [1], read $ua^2 + va^{\dagger 2}$ and $ua_1a_2 + va_1^\dagger a_2^\dagger$.

The Heisenberg–Weyl algebras h_1 and h_2, spanned by $1, a_1, a_1^\dagger$ and $1, a_1, a_1^\dagger, a_2, a_2^\dagger$ correspondingly, act irreducibly in the state spaces of one- and two-mode systems. The related families of CS $|\alpha\rangle$ and $|\alpha_1, \alpha_2\rangle$ are overcomplete and realize the continuous representations, which proved to be very efficient [10]. Therefore it is important to establish the relation between BG CS and the canonical CS representations. In the canonical CS representation every state $|\Psi\rangle$ is represented by an entire analytic function $F_{\rm CCS}(\alpha_1, \alpha_2; \Psi)$ of growth $(\frac{1}{2}, 2)$,

$$F_{\rm CCS}(\alpha_1, \alpha_2; \Psi) = \exp(\tfrac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2))\langle\alpha_1^*, \alpha_2^*|\Psi\rangle. \tag{10}$$

In the one-mode case $F_{\rm CCS}(\alpha) = \exp(\frac{1}{2}|\alpha|^2)\langle\alpha^*|\Psi\rangle$. The eigenvalue properties of the BG CS and canonical CS and the realizations (8) and (9) suggest that the canonical CS representation of a state $|\Psi\rangle \in \mathcal{H}_k$ should be obtained (up to a common factor) from its BG representation by means of substitution $z = \alpha^2/2$ for the one-mode system and $z = \alpha_1\alpha_2$ for the two-mode system, and this is the case. The corresponding relation between the two representations of the one-mode system states was written down in [2],

$$F_{\rm CCS}(\alpha; \Psi) = \pi^{\frac{1}{4}}\left[F_{\rm BG}\left(\frac{1}{2}\alpha^2, k = \frac{1}{4}\right) + \frac{1}{\sqrt{2}}\alpha F_{\rm BG}\left(\frac{1}{2}\alpha^2, k = \frac{3}{4}\right)\right]. \tag{11}$$

If $|\Psi\rangle$ is an even (odd) state, then the second (first) term is vanishing. For the two-mode system states the relation between F_{CCS} and F_{BG}, defined above, is found in the form (of proof in appendix A.1)

$$F_{\mathrm{CCS}}(\alpha_1,\alpha_2;\Psi)=F_{\mathrm{BG}}(z,k=\tfrac{1}{2};\Psi)+\sum_{k\geqslant 1}(\alpha_1^{2k-1}+\alpha_2^{2k-1})F_{\mathrm{BG}}(z,k;\Psi)\qquad z=\alpha_1\alpha_2. \tag{12}$$

Using these relations one can establish the coincidence between states, obtained in BG analytic representations and other familiar states. For example, the known one-mode even/odd CS $|\alpha\rangle_\pm$ coincide with the BG CS $|z;\frac{1}{4}\rangle$ and $|z;\frac{3}{4}\rangle$ [2, 27], while the generalized IS $|z,u,v;k\rangle$, constructed in [1] using BG representation, for $k=\frac{1}{4},\frac{3}{4}$ are the same as the eigenstates of $ua^2+va^{\dagger 2}$, constructed for real u,v in [25] and for complex u,v in [30, 4, 31, 9] using the canonical CS representation. For $k\geqslant\frac{1}{2}$ the $|z,u,v;k\rangle$ with real u,v can be identified with two-mode $SU(1,1)$ states of [16, 21, 22]. All $SU(1,1)$ states of [32, 33] can be found in the general family of $su^C(1,1)$ algebra related CS $|z,u,v,w;k\rangle$ constructed in [4, 6, 5].

In conclusion to this section it is worth noting that the $SU(1,1)$ group related CS [10] provides another analytic (in the unit disk) [28] representation of Hilbert space which has been shown [2] to be related to the BG representation through a Laplace transform. It is also worth making a note concerning the notation: the BG CS are eigenstates of the lowering operator $K_-=K_1-\mathrm{i}K_2$, which belongs to the complexified algebra $su^C(1,1)$. Therefore we could denote such states as $su^C(1,1)$ algebra related CS. However, usually when one deals with such simple complex combination as Weyl lowering/raising operators of an algebra L ($K_\pm$ for $su(1,1)$) one writes L instead of L^C ($su(1,1)$ instead of $su^C(1,1)$). For brevity we follow this convention for BG CS for Lie algebras. Continuous families of eigenstates of general elements of $su^C(1,1)$ have been considered and called $su^C(1,1)$ algebraic CS [4] or $SU(1,1)$ algebra eigenstates [5]. Another motivation of the new term 'algebra related CS' is the following property of the BG CS $|z;k\rangle$: unlike the h_n^C algebra CS this family cannot be represented in the form of group related CS either for the group $SU(1,1)$ or for the group of automorphysm $\mathrm{Aut}(su^C(1,1))\ni SU(1,1)$ [29].

3. The BG CS for $sp(N,R)$

The BG CS for semisimple Lie algebras can be naturally defined as eigenstates of mutually commuting Weyl lowering (or raising) operators $E_{\alpha'}$ ($E_{\alpha'}^\dagger$) [11]):

$$E_{\alpha'}|\boldsymbol{z}\rangle=z_{\alpha'}|\boldsymbol{z}\rangle. \tag{13}$$

This definition can be extended to any algebra, where lowering/raising operators exist. We shall consider here the simple Lie algebra $sp(N,R)$ (the symplectic algebra of rank N and dimension $N(2N+1)$). We redenote the Cartan–Weyl basis as $E_{ij},E_{ij}^\dagger,H_{ij}$ ($i,j=1,2,\ldots,N$, $E_{ij}=E_{ji}$, $H_{ij}^\dagger=H_{ji}$), and write the $sp(N,R)$ commutation relations

$$\begin{aligned}
&[E_{ij},E_{kl}]=[E_{ij}^\dagger,E_{kl}^\dagger]=0\\
&[E_{ij},E_{kl}^\dagger]=\delta_{jk}H_{il}+\delta_{il}H_{jk}+\delta_{ik}H_{jl}+\delta_{jl}H_{ik}\\
&[E_{ij},H_{kl}]=\delta_{il}E_{jk}+\delta_{jl}E_{ik}\\
&[E_{ij}^\dagger,H_{kl}]=-\delta_{ik}E_{jl}^\dagger-\delta_{jk}E_{il}^\dagger\\
&[H_{ij},H_{kl}]=\delta_{il}H_{kj}-\delta_{jk}H_{il}.
\end{aligned} \tag{14}$$

The BG CS $|\{z_{kl}\}\rangle$ for $sp(N,R)$ are defined as eigenstates of E_{ij},

$$E_{ij}|\{z_{kl}\}\rangle = z_{ij}|\{z_{kl}\}\rangle \qquad i,j = 1,2,\ldots,N. \tag{15}$$

Let us note that the Cartan subalgebra is spanned by H_{ii} only and $H_{i,j\neq i}$ are also Weyl lowering and raising operators as all E_{ij} are; we have simply separated the *mutually commuting lowering operators* E_{ij}. We shall construct explicitly the $sp(N,R)$ BG CS for the quadratic boson representation, which is realized by means of the operators

$$E_{ij} = a_i a_j \qquad E_{ij}^\dagger = a_i^\dagger a_j^\dagger \qquad H_{ij} = \tfrac{1}{2}(a_j^\dagger a_i + a_i a_j^\dagger) \tag{16}$$

where $a_i, a_i^\dagger$ are N pairs of boson annihilation and creation operators. These operators act irreducibly in the subspaces $\mathcal{H}^\pm$ spanned by the number states $|n_1,\ldots,n_N\rangle$ with even/odd $n_{\rm tot} \equiv n_1 + n_2 + \cdots + n_N$. The whole space $\mathcal{H}$ of the N-mode system is a direct sum of $\mathcal{H}^\pm$.

The $sp(N,C)$ is the complexification of $sp(N,R)$ and therefore the Hermitian quadratures of the above operators span over C the $sp(N,C)$ algebra. In the case of $N=1$ one obtains from (16) the three operators $K_{\pm,3}$ which close $sp(1,R) \sim su(1,1)$ (see equation (8)). We see that eigenstates of a^2 (the known even/odd states $|\alpha\rangle_\pm$ in quantum optics [14]) are $sp(1,R)$ BG CS for $k = \frac{1}{4}, \frac{3}{4}$.

One general property of $sp(N,R)$ CS $|\{z_{kl}\}\rangle$ for the representation (16) is that they depend effectively on N complex parameters α_j (not of N^2+N as one might expect). Indeed, using the boson commutation relations $[a_i,a_j]=0$ and the definition (15) we can easily derive

$$z_{ij}z_{kl} = z_{ik}z_{jl} = z_{il}z_{jk} \tag{17}$$

wherefrom we find the factorization of the eigenvalues z_{ij},

$$z_{ij} = \alpha_i\alpha_j \qquad \alpha_i,\alpha_j \in C. \tag{18}$$

Therefore in the above boson representation the definition (15) is rewritten as

$$a_i a_j|\{\alpha_k\alpha_l\}\rangle = \alpha_i\alpha_j|\{\alpha_k\alpha_l\}\rangle, \qquad i,j=1,2,\ldots,N. \tag{19}$$

The general solution to this system of equations is most easily obtained in the canonical CS representation [10]. In Dirac notations the solution reads

$$|\{\alpha_k\alpha_l\}; C_+, C_-\rangle = C_+(\boldsymbol{\alpha})|\boldsymbol{\alpha}\rangle + C_-(\boldsymbol{\alpha})|-\boldsymbol{\alpha}\rangle \equiv |\boldsymbol{\alpha}; C_+, C_-\rangle \tag{20}$$

where $|\boldsymbol{\alpha}\rangle$ are multimode canonical CS, $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\ldots,\alpha_N)$ and $C_\pm(\boldsymbol{\alpha})$ are arbitrary functions, subjected to the normalization condition ($|\boldsymbol{\alpha}|^2 = \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = |\alpha_1|^2 + \cdots + |\alpha_N|^2$)

$$|C_+(\boldsymbol{\alpha})|^2 + |C_-(\boldsymbol{\alpha})|^2 + 2\mathrm{Re}\,(C_-C_+^\dagger)N(|\boldsymbol{\alpha}|) = 1 \qquad N(|\boldsymbol{\alpha}|) = \langle\pm\boldsymbol{\alpha}|\mp\boldsymbol{\alpha}\rangle = \mathrm{e}^{-2|\boldsymbol{\alpha}|^2}. \tag{21}$$

Thus the families of states $|\boldsymbol{\alpha}; C_+, C_-\rangle$ represent *the whole set of* $sp(N,R)$ BG CS for the representation (16). They have the form of macroscopic superpositions of multimode canonical CS. The macroscopic superpositions of two canonical CS are also called Schrödinger cat states [14, 15], which we shall refer to as ordinary Schrödinger cat states. The set of (20) is the most general family of superpositions of the multimode CS $|\boldsymbol{\alpha}\rangle$ and $|-\boldsymbol{\alpha}\rangle$.

The large family of $sp(N,R)$ CS (20) contains many known, in quantum optics, subsets of states [14, 15] and many others not yet studied. Let us point out some of the well known particular subsets of (20). The limiting cases of $C_- = 0$ or $C_+ = 0$ recover the overcomplete family of multimode canonical CS, and $C_- = \pm C_+$ produces the ordinary multimode even/odd CS [15].

For the one-mode system ($N = 1$) several cases of the superpositions of two canonical CS (20) are thoroughly studied (for example, see [14] for $N = 1$ and [15] for any N). Nevertheless, as far as we know, even in the one-dimensional case no family of Schrödinger cat states was pointed out which is overcomplete in the strong sense in whole $\mathcal{H}$. Here we provide such families for any N.

Consider in (20) the choice of

$$C_+ = \cos\varphi \qquad C_- = \mathrm{i}\sin\varphi \tag{22}$$

which clearly satisfy the normcondition (21) for any angle φ,

$$|\boldsymbol{\alpha};\varphi\rangle = \cos\varphi|\boldsymbol{\alpha}\rangle + \mathrm{i}\sin\varphi|-\boldsymbol{\alpha}\rangle. \tag{23}$$

In the Fock basis (number states $|n_1,\ldots,n_N\rangle$) we have the expansion

$$|\boldsymbol{\alpha};\varphi\rangle = \mathrm{e}^{-|\boldsymbol{\alpha}|^2/2}\sum_{n_i=0}^{\infty}\frac{\alpha_1^{n_1}\ldots\alpha_N^{n_N}\mathrm{e}^{\mathrm{i}\varphi(-1)^{n_1+\cdots+n_N}}}{\sqrt{n_1!\ldots n_N!}}|n_1,\ldots,n_N\rangle. \tag{24}$$

Using direct calculations we find that these states resolve the unity operator for any φ and thereby provide an analytic representation in the whole $\mathcal{H}$,

$$1 = \frac{1}{\pi^N}\int \mathrm{d}^2\boldsymbol{\alpha}\,|\boldsymbol{\alpha};\varphi\rangle\langle\varphi;\boldsymbol{\alpha}| \qquad \mathrm{d}^2\boldsymbol{\alpha} = \mathrm{dRe}\,\alpha_1\,\mathrm{dIm}\,\alpha_1\ldots\mathrm{dRe}\,\alpha_N\,\mathrm{dIm}\,\alpha_N. \tag{25}$$

States $|\Psi\rangle$ are represented by functions

$$f_\Psi(\boldsymbol{\alpha},\varphi) = \mathrm{e}^{|\boldsymbol{\alpha}|^2/2}\langle\varphi,\boldsymbol{\alpha}^*|\Psi\rangle$$

on which the operators a_j and $a_j^\dagger$ act as

$$a_j = P_\varphi\alpha_j \qquad a_j^\dagger = P_\varphi\frac{\partial}{\partial\alpha_j} \tag{26}$$

where P_φ acts as an inversion operator with respect to φ: $P_\varphi f(\varphi) = f(-\varphi)$. At $\varphi = 0,\pi$ the multimode canonical CS representation $a_j = \alpha_j$, $a_j^\dagger = \partial/\partial\alpha_j$ is recovered.

The notation of (15) enables us to construct eigenstates of squared Weyl operators E_{ij}^2 (in any representation) as macroscopic superpositions of $sp(N,R)$ BG CS in the form (z_{ij} are eigenvalues of E_{ij})

$$|\{z_{kl}\};D_+,D_-\rangle = D_+(\{z_{ij}\})|\{z_{kl}\}\rangle + D_-(\{z_{ij}\})|\{-z_{kl}\}\rangle \tag{27}$$

where the functions $D_\pm(\{z_{ij}\})$ have to be subjected to the normalization condition (supposing $\langle\{z_{kl}\}|\{z_{kl}\}\rangle = 1$)

$$|D_+|^2 + |D_-|^2 + D_-D_+^*\langle\{z_{kl}\}|\{-z_{kl}\}\rangle + D_-^*D_+\langle\{-z_{kl}\}|\{z_{kl}\}\rangle = 1. \tag{28}$$

In the quadratic boson representation (16) these states take the form

$$\begin{aligned}|\{\alpha_k\alpha_l\};D_+,D_-\rangle &= D_+|\{\alpha_i\alpha_j\};C_+,C_-\rangle + D_-|\{-\alpha_i\alpha_j\};C_+,C_-\rangle\\ &\equiv |\boldsymbol{\alpha};C_+,C_-,D_+,D_-\rangle\end{aligned} \tag{29}$$

and can be termed *multimode squared amplitude Schrödinger cat states*. They are expected to exhibit linear and quadratic squeezing and other nonclassical properties. In view of (20) the states (29) are eventually expressed in terms of superpositions of four multimode canonical CS.

In conclusion to this section we note that the overcomplete family of states $|\boldsymbol{\alpha};\varphi\rangle$ admits n angles generalization: by means of n angles φ_k, $k = 1,2,\ldots,n$, n being a positive integer, one can construct macroscopic superpositions of 2^n CS $|\boldsymbol{\alpha}\rangle$ (or, equivalently, superpositions of 2^{n-1} $sp(N, R$ CS of the type $|\boldsymbol{\alpha};\varphi\rangle$), which are overcomplete and resolve the unity with respect to the same measure $\pi^{-N}\,\mathrm{d}^2\boldsymbol{\alpha}$,

$$|\boldsymbol{\alpha};\varphi_1,\ldots,\varphi_n\rangle = \cos\varphi_n|\boldsymbol{\alpha};\varphi_1,\ldots,\varphi_{n-1}\rangle + \mathrm{i}\sin\varphi_n|-\boldsymbol{\alpha};\varphi_1,\ldots,\varphi_{n-1}\rangle \tag{30}$$

$$1 = \frac{1}{\pi^N}\int \mathrm{d}^2\boldsymbol{\alpha}\,|\boldsymbol{\alpha};\varphi_1,\ldots,\varphi_n\rangle\langle\varphi_n,\ldots,\varphi_1;\boldsymbol{\alpha}|. \tag{31}$$

In every component state in $|\boldsymbol{\alpha};\varphi_1,\ldots,\varphi_n\rangle$ the parameters α_i are on a circle with radius $|\alpha_i|$. For $n=0$ we have CS $|\boldsymbol{\alpha}\rangle$, for $n=1$ the states (23) are reproduced. $|\boldsymbol{\alpha};\varphi_1,\ldots,\varphi_n\rangle$ are easily seen to be eigenvectors of $(a_ia_j)^{2^{n-1}}$, and not of $(a_ia_j)^m$, $m<2^{n-1}$, unless φ_k are integer multiples of $\pi/2$. In the one mode case ($N=1$) $|\alpha;\varphi_1,\ldots,\varphi_n\rangle$ are eigenstates of a^{2^n}. Eigenstates of a^{2k} for $k=1,2,\ldots,$ can be easily constructed as superpositions of $sp(1,R)$ CS. Here we proved their overcompleteness for $2k=2^n=2,4,8,32\ldots.$ Some eigenstates of powers of a^k, $k>2$, have been considered in [34]. Multicomponent macroscopic superpositions of canonical CS (one mode only so far) are intensively studied in quantum optics (with the final aim being the production of Fock states) [35–37].

The above result (31) is a particular case of a general theorem, proved in appendix A.2, concerning the overcompleteness of common eigenstates of powers of N non-Hermitian operators $A_j^{2^n}$, $j=1,\ldots,N$, $n=1,2,\ldots,$ and valid for the case of N-mode canonical CS and also for $sp(N,R)$ BG CS.

4. BG CS for the algebra $u(p,q)$

The algebras $u(p,q)$, $p+q=N$, are real forms of $sl(N,C)$ and they are subalgebras of $sp(N,R)$ [11]. Therefore the BG CS for $u(p,q)$ should be obtained from $sp(N,R)$ CS by a suitable restriction. In this section we consider these problems in greater detail in the boson representation (16).

The following subset of operators of (16) close the $u(p,q)$ algebra (or $u^C(p,q)$ if one considers non-Hermitian linear combinations of the operators below) [11],

$$E_{\alpha\mu}=a_\alpha a_\mu \qquad E^\dagger_{\alpha\mu}=a^\dagger_\alpha a^\dagger_\mu \qquad H_{\alpha\beta}=\tfrac12(a^\dagger_\alpha a_\beta + a_\beta a^\dagger_\alpha) \qquad H_{\mu\nu}=\tfrac12(a^\dagger_\mu a_\nu + a_\nu a^\dagger_\mu) \tag{32}$$

where we adopted the notations $\alpha,\beta,\gamma=1,\ldots,p$, $\mu,\nu=p+1,\ldots,p+q$, $p+q=N$ (while $i,j,k,l=1,2,\ldots,N$). For $p=1=q$ the three standard $su(1,1)$ operators $K_\pm, K_3$ are $K_-=E_{12}=a_1a_2$, $K_+=E^\dagger_{12}=a^\dagger_1a^\dagger_2$, $K_3=(a^\dagger_1a_1+a^\dagger_2a_2+1)/2$. The subsets of Hermitian operators

$$\begin{aligned} M^{(p)}_{\alpha\beta} &= \tfrac12(H_{\alpha\beta}+H_{\beta\alpha}-\delta_{\alpha\beta}) \qquad & \tilde{M}^{(p)}_{\alpha\beta} &= \mathrm{i}(H_{\beta\alpha}-H_{\alpha\beta})\\ M^{(q)}_{\mu\nu} &= \tfrac12(H_{\mu\nu}+H_{\nu\mu}-\delta_{\mu\nu}) \qquad & \tilde{M}^{(q)}_{\mu\nu} &= \mathrm{i}(H_{\nu\mu}-H_{\mu\nu}) \end{aligned} \tag{33}$$

realize representations of compact subalgebras $u(p)$ and $u(q)$ correspondingly. The $u(p,q)$ algebra (32) acts irreducibly in the subspaces of eigenstates of the Hermitian operator L,

$$L=\sum_\alpha M^{(p)}_{\alpha\alpha}-\sum_\mu M^{(q)}_{\mu\mu}=\sum_\alpha H_{\alpha\alpha}-\sum_\mu H_{\mu\mu}-(p-q)/2. \tag{34}$$

This is the linear-in-generators Casimir operator and the higher Casimirs here are expressed in terms of L [13]. Denoting the eigenvalue of L by l we have the expansion $\mathcal{H}=\sum_{l=-\infty}^{\infty}\oplus\mathcal{H}_l$. The representations corresponding to $\pm l$ are equivalent (but the subspaces $\mathcal{H}_{\pm l}$ are orthogonal). We note that $L=\sum_\alpha a^\dagger_\alpha a_\alpha - \sum_\mu a^\dagger_\mu a_\mu$, and $l=0,\pm1,\ldots.$

The commuting Weyl lowering operators of $u(p,q)$ are $E_{\mu\gamma}=a_\mu a_\gamma$, $\gamma=1,2,\ldots,p$, $\mu=p+1,p+2,\ldots,p+q=N$. We have proved in the above that eigenvalues of the product of two boson destruction operators are factorized. Therefore the $u(p,q)$ BG CS in the above boson representation can be defined as $|\{\alpha_\beta\alpha_\nu\};l,p,q\rangle$,

$$a_\mu a_\gamma|\{\alpha_\beta\alpha_\nu\};l,p,q\rangle = \alpha_\mu\alpha_\gamma|\{\alpha_\beta\alpha_\nu\};l,p,q\rangle \qquad \gamma = 1,\ldots,p \qquad \mu = p+1,\ldots,p+q \tag{35}$$

where α_μ and α_γ are arbitrary complex numbers. We put $\|\boldsymbol{\alpha};l,p,q\rangle = \|\{\alpha_\beta\alpha_\nu\};l,p,q\rangle$, denoting by $\|\Psi\rangle$ a non-normalized (but normalizable) state, while $|\Psi\rangle$ is normalized to unity. Solutions to the above equations can be written in the form

$$\|\boldsymbol{\alpha};l,p,q\rangle = \sum_{\tilde n_p-\tilde n_q=l}\frac{\alpha_1^{n_1}\ldots\alpha_{N-1}^{n_{N-1}}\alpha_N^{\tilde n_p-\tilde n'_q-l}}{\sqrt{n_1!\ldots n_{N-1}!(\tilde n_p-\tilde n'_q-l)!}}|n_1,\ldots,n_{N-1};\tilde n_p-\tilde n'_q-l\rangle \tag{36}$$

where α_i, $i=1,\ldots,N$, are arbitrary complex parameters, $\tilde n_p = \sum_\alpha n_\alpha$, $\tilde n_q = \sum_\mu n_\mu$, $\tilde n'_q = \tilde n_q - n_N$ and $l = \tilde n_p - \tilde n_q$. In (36) summation is over all $n_i = 0,1,2,\ldots$ provided $\tilde n_p - \tilde n_q = l =$ constant.

If we multiply $\|\boldsymbol{\alpha};l,p,q\rangle$ by $\exp(-|\boldsymbol{\alpha}|^2/2)$ and sum over l we evidently obtain the normalized multimode CS $|\boldsymbol{\alpha}\rangle$ (for any pair p,q),

$$|\boldsymbol{\alpha}\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{l=-\infty}^{\infty}\|\boldsymbol{\alpha};l,p,q\rangle. \tag{37}$$

The last equality suggests that the states $\|\boldsymbol{\alpha};l,p,q\rangle$ form overcomplete families in $\mathcal{H}_l$ for every p,q. This is the case: using the overcompleteness of $|\boldsymbol{\alpha}\rangle$, formula (37) and the orthogonality relations

$$\langle p,q,l';\boldsymbol{\alpha}\|\boldsymbol{\alpha};l,p,q\rangle = 0 \qquad \text{for } l'\neq l \tag{38}$$

one obtains the resolution of unity in $\mathcal{H}_l$ in terms of the $u(p,q)$ CS $\|\boldsymbol{\alpha};l,p,q\rangle$,

$$\int \mathrm{d}\mu\,(\boldsymbol{\alpha})\|\boldsymbol{\alpha};l,p,q\rangle\langle p,q,l;\boldsymbol{\alpha}\| = 1_l \qquad \mathrm{d}\mu\,(\boldsymbol{\alpha}) = \frac{1}{\pi^N}\mathrm{e}^{-|\alpha|^2}\,\mathrm{d}^2\boldsymbol{\alpha}. \tag{39}$$

Now we note that in $u(p,q)$ CS (36) one complex parameter, say α_N, can be absorbed into the normalization factor by redefining the rest as

$$z_1 = \alpha_1\alpha_N,\ldots,z_p = \alpha_p\alpha_N \qquad z_{p+1} = \alpha_{p+1}/\alpha_N,\ldots,z_{N-1} = \alpha_{N-1}/\alpha_N. \tag{40}$$

Then we can write $\|\boldsymbol{\alpha};l,p,q\rangle = \alpha_N^{-l}\|\boldsymbol{z};l,p,q\rangle$ and

$$\|\boldsymbol{z};l,p,q\rangle = \sum_{\tilde n_p-\tilde n_q=l}\frac{z_1^{n_1}\ldots z_{N-1}^{N-1}}{\sqrt{n_1!\ldots n_{N-1}!(\tilde n_p-\tilde n'_q-l)!}}|n_1,\ldots,n_{N-1};\tilde n_p-\tilde n'_q-l\rangle \tag{41}$$

where $\boldsymbol{z} = (z_1,\ldots,z_{N-1})$. The states $\|\boldsymbol{z};l,p,q\rangle$ are normalizable in view of

$$1 = \langle\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle = \mathrm{e}^{-|\alpha|^2}\sum_{l=-\infty}^{\infty}|\alpha_N|^{-2l}\langle q,p,l;\boldsymbol{z}\|\boldsymbol{z};l,p,q\rangle$$

which stems from (37) and (38). The normalized states $|\boldsymbol{z};l,p,q\rangle$ are $|\boldsymbol{z};l,p,q\rangle = \mathcal{N}\|\boldsymbol{z};l,p,q\rangle$, $\mathcal{N}$ being the normalization constant.

The family $\{\|\boldsymbol{z};l,p,q\rangle\}$ is overcomplete in $\mathcal{H}_l$ and the resolution of unity reads $(\mathrm{d}^2\boldsymbol{z} = \prod_i^{N-1}\mathrm{dRe}\,z_i\,\mathrm{dIm}\,z_i = |\alpha_N|^{2(q-1-p)}\prod_i^{N-1}\mathrm{dRe}\,\alpha_i\mathrm{dIm}\,\alpha_i)$

$$1_l = \int \mathrm{d}\mu\,(\boldsymbol{z};l,p,q)\|\boldsymbol{z};l,p,q\rangle\langle q,p,l;\boldsymbol{z}\| \qquad \mathrm{d}\mu(\boldsymbol{z},l,p,q) = F(|\tilde{\boldsymbol{z}}_p|,|\tilde{\boldsymbol{z}}_q|;l,p,q)\,\mathrm{d}^2\boldsymbol{z} \tag{42}$$

where $|\tilde{z}_p|^2 = |z_1|^2 + \cdots + |z_p|^2$, $|\tilde{z}_q|^2 = |z_{p+1}|^2 + \cdots + |z_{N-1}|^2$, the measure weight function being

$$F(|\tilde{z}_p|, |\tilde{z}_q|; l, p, q) = \frac{1}{\pi^N} \int \mathrm{d}^2\alpha_N\, |\alpha_N|^{2(q-1-p-l)} \times \exp\left[-\left(\frac{|\tilde{z}_p|^2}{|\alpha_N|^2} + |\tilde{z}_q|^2|\alpha_N|^2 + |\alpha_N|^2\right)\right]. \tag{43}$$

One can prove that the above measure is unique in the class of smooth functions of $|z_1|, \ldots, |z_{N-1}|$ (see appendix A.3). Thus the explicit form of $u(p,q)$ BG CS is

$$|\boldsymbol{z}; l, p, q\rangle = \mathcal{N}(|z_1|, \ldots, |z_{N-1}|; l, p, q)\|\boldsymbol{z}; l, p, q\rangle \tag{44}$$

where $\|\boldsymbol{z}; l, p, q\rangle$ take the form of superposition (41) of multimode Fock states with fixed value l of the difference number operator L, equation (34).

Let us note some known particular cases of the $u(p,q)$ BG CS (41). Recently the case of $q = 1$ and negative l, $-l \geqslant 0$ (then $p = N-1$, $\tilde{n}'_q = 0$, $\boldsymbol{z}_q = 0$ and $\boldsymbol{z}_p \equiv \boldsymbol{z}$) has been considered by Fujii and Funahashi [7]. Their resolution unity measure (in $\mathcal{H}_l$) reads

$$\mathrm{d}\mu'(\boldsymbol{z}) = F'(|\boldsymbol{z}|, l, p, 1)d^2\boldsymbol{z} \qquad F' = \frac{2|\boldsymbol{z}|^{-l-p+1}}{\pi^p} K_{-l-p+1}(2|\boldsymbol{z}|) \tag{45}$$

where $K_\nu(z)$ is the modified Bessel function of the third kind [26]. $F'(|\boldsymbol{z}|, l, p, 1)$ and $F(|\boldsymbol{z}|, l, p, 1)$ do not depend on phases of z_i and are smooth functions of $|z_1|, \ldots, |z_p|$, i.e. all order derivatives are finite. In appendix A.3 we prove that the resolution unity measures for $u(p,q)$ CS are unique within such a class of functions, i.e. $F'(|\boldsymbol{z}|, l, p, 1)$ and $F(|\boldsymbol{z}|, l, p, 1)$ should coincide. Then using the analyticity property of Bessel functions $K_\nu(z)$ [26] we establish (cf proof in appendix A.4) the following integral representation for $K_\nu(z)$ with $\nu = 0, \pm 1, \ldots$ and $\mathrm{Re}\, z \geqslant 0$

$$K_\nu(2z) = \tfrac{1}{2} z^{-\nu} \int_0^\infty \mathrm{d}x\, x^{\nu-1} \mathrm{e}^{-(x+z^2/x)}. \tag{46}$$

For $p = 1, q = 1$ our states $|\boldsymbol{z}; l, p, q\rangle$ recover (as the states of [7] do) the BG CS $|z; k\rangle$ for the series $D^+(k)$ of $su(1,1)$ [8], the Bargman index k being expressed in terms of l as $k = (1 + |l|)/2$. The irreps with $\pm l$ are equivalent, however, the states $|z; l, 1, 1\rangle$ and $|z; -l, 1, 1\rangle$ are different as one can see from their definition (41) (moreover, they are orthogonal). Thus our states $|z; \pm l, 1, 1\rangle$ represent two equivalent but different realizations of BG CS $|z; k\rangle$ for $k = (1 + |l|)/2 = \frac{1}{2}, 1, \ldots.$ The exact identification is $\|z; l \leqslant 0, 1, 1\rangle = \|z; k\rangle$, $\|z; l > 0, 1, 1\rangle = z^{2k-1}\|z; k\rangle$.

The pair of CS in quantum optics $|\zeta, q\rangle$ [16] (defined as eigenstates of a_1a_2 with $a_1^\dagger a_1 - a_2^\dagger a_2 = q = \text{constant}$) appear as $u(1,1)$ BG CS $|z; k\rangle$ in the two-mode representation ($N = 2$ in (32)). The identifications are $|\zeta, q\rangle = |z; l, 1, 1\rangle$, i.e. the Agarwal ζ and q are equal to our z and l correspondingly. In view of equations (42)–(45) the pair of CS are overcomplete in the subspaces $\mathcal{H}_l$. Our $|\boldsymbol{z}; l, p, q\rangle$ can be regarded as a generalization of $|\zeta, q\rangle$ to the N-mode boson system: $|\boldsymbol{z}; l, p, q\rangle$ are invariant under the annihilation of pairs of two different mode bosons, one from the first p modes, and the other from the last q modes. Note that in the $sp(N,R)$ CS $|\boldsymbol{\alpha}, C_-, C_+\rangle$ there is no such restriction—these are the most general states, which are invariant under the annihilation of any pair of bosons.

The $sp(N,R)$ CS $|\boldsymbol{\alpha}, C_-, C_+\rangle$ can be decomposed in terms of $u(p,q)$ CS $|\boldsymbol{z}; l, p, q\rangle$ with different l. For $N = 2$ this decomposition reads

$$\begin{aligned} &|\boldsymbol{\alpha}; C_-, C_+\rangle = \sum_{l=0}^\infty \alpha_1^l \tilde{C}_l |z; -l, 1, 1\rangle + \sum_{l=1}^\infty \alpha_2^l \tilde{C}_l |z; l, 1, 1\rangle \\ &\tilde{C}_l = C_+ + (-1)^l C_- \qquad z = \alpha_1\alpha_2. \end{aligned} \tag{47}$$

In analogy with the case of $sp(N,R)$, considered in section 3 we introduce the $u(p,q)$ multimode squared amplitude cat states $|\boldsymbol{z}; l, p, q; D_+, D_-\rangle$ as macroscopic superpositions of $u(p,q)$ BG-type CS $|\boldsymbol{z}; l, p, q\rangle$,

$$|\boldsymbol{z}; l, p, q; D_+, D_-\rangle = D_+|\boldsymbol{z}; l, p, q\rangle + D_-|-\boldsymbol{z}; l, p, q\rangle \tag{48}$$

which are expected to exhibit squared amplitude squeezing and other nonclassical properties. In the particular cases of $N = 2$, $D_- = D_+ \exp(\mathrm{i}\phi)$ the states (48) recover the two-mode Schrödinger cat states, considered recently in [20].

5. Statistical properties of the N-mode $sp(N,R)$ BG CS and their macroscopic superpositions

In this section we consider some general statistical properties of the constructed $sp(N,R)$ algebra related CS and their superpositions and discuss in greater detail some new subsets of this large family.

All $sp(N,R)$ BG-type CS minimize the Robertson multidimensional uncertainty relation [38] for the Hermitian quadratures X_{ij}, Y_{ij} of mutually commuting Weyl lowering operators E_{ij}, since they are eigenstates of all E_{ij} (proposition 3 of [6]),

$$\det\sigma(\{X_{ij}, Y_{ij}\}; \boldsymbol{\alpha}, C_-, C_+) = \det C(\{X_{ij}, Y_{ij}\}; \boldsymbol{\alpha}, C_-, C_+) \tag{49}$$

where σ is the matrix of second moments of all of the observables X_{ij}, Y_{ij} (the uncertainty matrix) and C is the antisymmetric matrix of all mean commutators of X_{ij}, Y_{ij} times $(-i/2)$. The number of commuting E_{ij} is equal to $(N^2 + N)/2$. Robertson inequality for n observables X_j, $j = 1, 2, \ldots, n$, reads

$$\det\sigma(\{X_j\}; \Psi) \geqslant \det C(\{X_j\}; \Psi) \tag{50}$$

and for a pair of operators X, Y it reduces to the Schrödinger case, $\Delta^2 X \Delta^2 Y - \sigma_{XY}^2 \geqslant |\langle[X,Y]\rangle|^2/4$, where $\sigma_{XX} = \langle XY + YX\rangle/2 - \langle X\rangle\langle Y\rangle$ (for greater detail see for example, [1, 6]). In all $sp(N,R)$ BG CS the covariances of X_{ij} and Y_{ij} are vanishing, but those of X_{ij} and X_{kl} are not, i.e. the matrix $\sigma(\{X_{ij}, Y_{ij}\}; \boldsymbol{\alpha}, C_-, C_+)$ is not diagonal.

The subset of Schrödinger cats $|\boldsymbol{\alpha}, \varphi\rangle$, equation (23), possess several remarkable properties:

(a) they are overcomplete in the whole Hilbert space (see equation (25));

(b) the photon statistics in every mode is Poissonian for any $\boldsymbol{\alpha}$ and φ. This follows immediately from the expansion (24) in terms of multimode number states $|\boldsymbol{n}\rangle = |n_1, \ldots, n_N\rangle$;

(c) the states $|\boldsymbol{\alpha}; \varphi\rangle$ can exhibit squeezing in the quadratures p_j, q_j (for example, for $|\boldsymbol{\alpha}|$ close/equal to $|\alpha_i| = 0.5$, $\phi = \pi/4$ and $\arg\alpha_i$ around $n\pi/2, n = 0, 1, \ldots$, the minimal value of Δp_i and Δq_i being equal to 0.316—see the graphics f_1 on figure 1);

(d) these states are physically coherent (‘true coherent’) since they satisfy the condition of full second-order coherence of the field [39]. The latter property again follows from equation (24), which is of the form of generalized CS of Glauber and Titulaer [39].

This interesting subfamily $\{|\boldsymbol{\alpha}; \varphi\rangle\}$ of $sp(N,R)$ BG CS can be generated from the familiar multimode canonical CS by means of the following operator

$$S(\varphi) = \exp(\mathrm{i}(-1)^{\hat{n}}\varphi) \qquad |\boldsymbol{\alpha}, \varphi\rangle = S(\varphi)|\boldsymbol{\alpha}\rangle \tag{51}$$

where $\hat{n} = a_1^\dagger a_1 + \cdots + a_N^\dagger a_N$ is the total number operator. As strange as it may seem $S(\varphi)$ is well defined for any angle φ and is unitary. On any state $|\Psi\rangle$ its action is

$$S(\varphi)|\Psi\rangle = \mathrm{e}^{\mathrm{i}\varphi}\|\Psi\rangle_e + \mathrm{e}^{-\mathrm{i}\varphi}\|\Psi\rangle_o \tag{52}$$

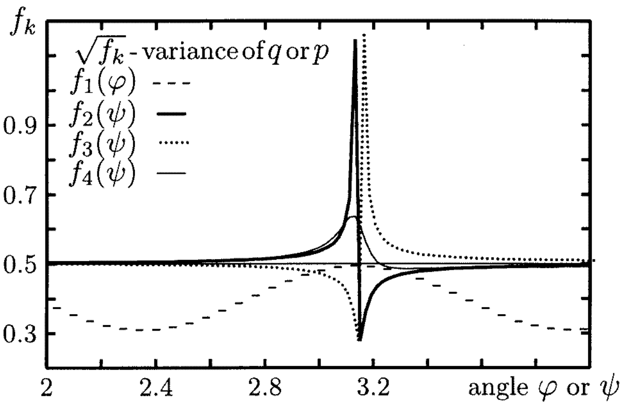

Figure 1. Amplitude quadrature squeezing in the $sp(N, R)$ BG CS $|\alpha;\varphi\rangle$ and the superpositions $|\alpha,\phi,\psi\rangle$, equation (61), for $r_i = |\alpha_i| = 0.05$. $\Delta\tilde{q}_i = \Delta\tilde{q}_i(\tilde{r}, r_i, \theta_i, \phi, \psi)$, $\tilde{r} = |\alpha|$, $\tilde{q}_i = q_i$ or p_i. $f_1 = \Delta^2 q_i(r_i, r_i, -\frac{\pi}{2}, \varphi)$, $f_2 = \Delta^2 p_i(r_i, r_i, \frac{\pi}{4}, 0, \psi)$, $f_3 = \Delta^2 q_i(r_i, r_i, \frac{\pi}{4}, 0, \psi)$, $f_4 = \Delta^2 p_i(4r_i, r_i, \frac{\pi}{4}, 0, \psi)$. $|\alpha,\varphi\rangle$ are weakly nonclassical states for every mode, $|\alpha,\phi,\psi\rangle$ are strongly nonclassical.

where $\|\Psi\rangle_{e,o}$ are the projections of $|\Psi\rangle$ on even/odd subspaces $\mathcal{H}^{\pm}$. The operator $(-1)^{\hat{n}}$ is Hermitian and $(-1)^{\hat{n}}\varphi$ may be regarded as a sort of nonlinear multimode interaction.

In the classification scheme of [18] the (one-mode) states which possess the above properties (b) and (c) fall into the subclass of the *weakly nonclassical* states. In this scheme the nonclassical states are subdivided into *weakly nonclassical* and *strongly nonclassical* depending on the pointwise non-negativity or nonpositivity of the phase averaged $\mathcal{P}(I)$ Glauber–Sudarshan diagonal representation $P(\beta)$, $I = |\beta|^2$, $\beta = \sqrt{I}\exp(\mathrm{i}\vartheta)$,

$$\mathcal{P}(I) = \frac{1}{2\pi}\int_0^{2\pi} P(r\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}^2\vartheta \qquad r = \sqrt{I} = |\beta|. \tag{53}$$

If $\mathcal{P}(I) < 0$ for some values of I the state is strongly nonclassical (then also $P(\beta) < 0$ for some values of β) and if $\mathcal{P}(I) \geqslant 0$ but $P(\beta) \not\geqslant 0$ the state is said to be weakly nonclassical [18]. The set of classical states (i.e. $P(\beta) \geqslant 0$) is not subdivided. Criteria for phase-insensitive nonclassicality of single-mode states were also studied in [19].

The family of $sp(N, R)$ BG CS $|\alpha;\varphi\rangle$, equation (23), consists of classical (at $\varphi = 0, \pm\pi/2, \pi$) and weakly nonclassical states (for $\varphi \neq 0, \pm\pi, \pi$) for every mode since the multimode photon distribution in these states is a product of one-mode Poisson distributions. There are no strongly nonclassical states in this family. Note that at $\varphi = 0, \pm\pi/2, \pi$ the states $|\alpha;\varphi\rangle$ are the CS $|\alpha\rangle$ or $|-\alpha\rangle$, and at $\varphi = \pi/4$ they coincide with the Yurke–Stoler states [14]. The states with Gaussian Wigner function are either classical or strongly nonclassical [18] and strongly nonclassical states from the latter family all have a positive Mandel Q factor (super-Poissonian statistics) [40, 41] [$Q = (\Delta^2\hat{n} - \langle\hat{n}\rangle)/\langle\hat{n}\rangle$, where $\hat{n} = a^\dagger a$].

Along these lines we note that in the family of weakly nonclassical states $|\alpha;\varphi\rangle$ there are states which exhibit quadrature squeezing (graphics f_1 on figure 1). Conversely, there exist strongly nonclassical states (for example, in the family $|\alpha,\phi,\psi\rangle$, defined below) which do not exhibit squeezing of the quadratures of either a or a^2 (nor is the Q factor negative). Moreover, among the one-mode $|\alpha,\phi,\psi\rangle$ there are states with $Q = 0$ which are squeezed or not squeezed, but their photon statistics are not Poissonian (see the graphics on figures 4 and 5). These examples show that $Q = 0$ is not a sufficient condition either for statistics to be Poissonian or for a state to be classical.

The quadrature squeezing and/or $Q < 0$ are sufficient conditions for the nonclassicality of the corresponding states [41]. However, they are neither sufficient nor necessary for the strong nonclassicality as demonstrated below.

In [18] a simple sufficient condition for strong nonclassicality of the states (i.e. for nonpositivity of the phase smeared diagonal P representation $\mathcal{P}(I)$) is given in terms of photon number distributions p_n,

$$l_n := (n+1)p_{n-1}p_{n+1} - np_n^2 < 0 \qquad \text{for some } n > 0. \tag{54}$$

The distribution p_n is expressed in terms of $\mathcal{P}(I)$ as [18]

$$p_n = \int_0^\infty \mathrm{d}I\,\mathcal{P}(I)\,p_n{}^{(\mathrm{Pois})}(I) \qquad p_n{}^{(\mathrm{Pois})}(I) = \frac{1}{n!}I^n \mathrm{e}^{-I}. \tag{55}$$

Distributions p_n which can be represented in the above form with $\mathcal{P} \geqslant 0$ ($\mathcal{P} \ngeqslant 0$) were recently defined as classical (nonclassical) [24]. Nonclassicality of p_n means strong nonclassicality of the corresponding states.

Among $sp(N,R)$ BG CS there are strongly nonclassical states as well (the definition of strong nonclassicality for multimode states is discussed below), such as, for example, the cat states

$$|\boldsymbol{\alpha},\phi\rangle = \tilde{\mathcal{N}}(|\boldsymbol{\alpha}\rangle + \mathrm{e}^{\mathrm{i}\phi}|-\boldsymbol{\alpha}\rangle) \tag{56}$$

the normalization constant being

$$\tilde{\mathcal{N}} = (2(1+\cos\phi\,\mathrm{e}^{-2\tilde{r}^2}))^{-\frac{1}{2}} = \tilde{\mathcal{N}}(\tilde{r},\phi).$$

In the case of $N = 1$ the states (56) have been discussed, for example in [9, 24] and in the fifth paper of [14]. The probability for totally n photons in $|\boldsymbol{\alpha},\phi\rangle$ (irrespective of which mode they belong to), $n = n_1 + \cdots + n_N$, is found as

$$p_n(\tilde{r},\phi) = \tilde{N}(\tilde{r},\phi)^2 \mathrm{e}^{-\tilde{r}^2}\frac{\tilde{r}^{2n}}{n!}s_n(\phi) \qquad s_n(\phi) = 2(1+(-1)^n\cos\phi) \qquad \tilde{r} = |\boldsymbol{\alpha}| \tag{57}$$

and the function $l_n(\tilde{r},\phi)$ takes the form

$$l_n(\tilde{r},\phi) = \tilde{l}_n(\tilde{r},\phi)(s_{n-1}(\phi)s_{n+1}(\phi) - s_n^2(\phi)) \tag{58}$$

where the function $\tilde{l}_n$,

$$\tilde{l}_n = \tilde{N}(\tilde{r},\phi)^2 \mathrm{e}^{-\tilde{r}^2}\frac{\tilde{r}^{4n}}{n!(n-1)!}$$

is non-negative. The non-negative factor $s_n(\phi)$ is seen to be a bounded and oscillating function of both ϕ and n, $s_n(\phi) = s_{n+2}(\phi)$. Then for every $\phi \neq \pm\pi/2$ the combination $s_{n-1}(\phi)s_{n+1}(\phi) - s_n^2(\phi)$ is negative for all n for which $(-1)^n\cos\phi > 0$. Noting that the total photon number distribution $p_n(\tilde{r},\phi)$, equation (57), coincides with that for the one-mode states $|\tilde{\alpha},\phi\rangle$, $|\tilde{\alpha}| = \tilde{r}$, we conclude that all one-mode states $|\alpha,\phi \neq \pm\pi/2\rangle$ are strongly nonclassical (the strong nonclassicality of the one-mode states $|\alpha,\phi\rangle$ was also proved in the very recent E-print [24]).

One way of generalizing the notion of strong nonclassicality to multimode states is to apply the above definition to the total photon number distribution, the other is to require this for every mode. One can easily verify, that in the N-mode states $|\boldsymbol{\alpha},\phi\rangle$ the conditional photon distributions $p_{n_1,\ldots,n_i,\ldots,n_N}(\boldsymbol{\alpha},\phi)$ for the individual mode i ($n_{k\neq i}$ being fixed) also obey the inequality (54). Thus $|\boldsymbol{\alpha},\phi \neq \pm\pi/2\rangle$ are strongly nonclassical according to both criteria.

Now consider the *squared amplitude quadrature squeezing* [17] in the multimode states. We first note that the BG-type CS for any Lie algebra cannot exhibit squeezing of the quadratures X_{ij} and Y_{ij} of Weyl operators E_{ij} since here the variances of X_{ij} and Y_{ij} are equal which stems from their eigenvalue property (15) [1]. In the quadratic boson representation $E_{ij} = a_i a_j$ and X_{ij} (or Y_{ij}) squeezing is multimode squared amplitude squeezing. The quadrature X_{ij} (Y_{ij}) of $a_i a_j$ is said to be squeezed in a state $|\Psi\rangle$ if the variance ΔX_{ij} (ΔY_{ij}) is less than its value in the ground state $|\mathbf{0}\rangle$. Thus quadratic field squeezing does not occur in $|\boldsymbol{\alpha}; C_-, C_+\rangle$. We shall see that macroscopic superpositions of two such states do exhibit quadratic squeezing. However, let us first make some general remarks about the SS of two and several observables.

Squeezing of the two quadratures X and Y of a non-Hermitian operator A (for definiteness we write $A = X + \mathrm{i}Y$) can be achieved in two ways:

(a) in the eigenstates $|z, u, v\rangle$ of complex combination $uA + vA^\dagger$ (generalized IS) [1];

(b) in the eigenstates $|z\rangle^{(2)}$ of A^2 (generalized cat states).

The first possibility was proved and demonstrated (in the examples of $SU(1,1)$ and $SU(2)$ generators in the series $D^+(k)$ and $D(j)$) in [1, 4]. These SS minimize the Schrödinger inequality and therefore were called Schrödinger (or generalized) IS. A particular case of the SS of type (a) are the SS for general systems [25], introduced as states minimizing the Heisenberg inequality, which is a particular case of that of Schrödinger. The second possibility (b) can be proved easily by calculations using the eigenvalue condition of A^2 and taking into account the Schrödinger relation. This can also be checked directly on the example of the following two types of superposition states

$$|z; \varphi\rangle = \cos\varphi|z\rangle + \mathrm{i}\sin\varphi|-z\rangle \qquad |z, \phi\rangle = \mathcal{N}(|z\rangle + \mathrm{e}^{\mathrm{i}\phi}|-z\rangle) \tag{59}$$

where $|z\rangle$ are eigenstates of A, $A|\pm z\rangle = \pm z|\pm z\rangle$. These states can exhibit squeezing according to the stronger criterion, given in [1] (see also below). $|z;\varphi\rangle$ and $|z,\phi\rangle$ are eigenstates of A^2 (and not of A, unless $\varphi = n\pi/2$, $n = 0, 1, \ldots$). Eigenstates $|z\rangle^{(2)}$ of A^2 which are not eigenstates of A are superpositions of $|\pm z\rangle$. Therefore the SS of type (b) are cat states. $|z;\varphi\rangle$ and $|z,\phi\rangle$ in (59) are examples of such SS for any A for which eigenstates $|\pm z\rangle$ do exist. In fact first- and higher order squeezing of X and Y can occur in states which are eigenvectors of A^n for any $n \geqslant 1$ and such eigenvectors can be easily expressed as discrete superpositions of several $|z\rangle$.

The operator $S(u, v)$ which transforms the nonsqueezed $|z\rangle$ to the SS of type (a), $|z, u, v\rangle$, was defined in [1, 6] as a generalized squeeze operator (if $A = a$ then $S(u, v)$ is the known canonical squeeze operator [41, 42]). Having established that eigenstates $|z\rangle^{(2)}$ of A^2 can universally exhibit squeezing of the quadrature of A we can define, in analogy with the previous case, a *squeeze operator of the second kind* S_{II} by means of the relation

$$|z\rangle^{(2)} = S_{II}|z\rangle. \tag{60}$$

We can point out an example of such a squeeze operator—that is the operator $S(\varphi)$ of equation (51). It maps the multimode CS $|\boldsymbol{\alpha}\rangle$ to the weakly nonclassical states ($sp(N, R)$ BG CS) $|\boldsymbol{\alpha}; \varphi\rangle$ which are eigenstates of $a_i a_j$ (the $sp(N, R)$ BG CS) and do exhibit quadrature squeezing (see the graphics f_1 on figure 1).

The main difference between the above two types of SS is the following. SS of type (a) can exhibit arbitrary strong squeezing of X or Y, while the squeezing in SS of type (b) is always bounded, since eigenstates of $A^2 \sim (X + \mathrm{i}Y)^2$ can never tend to an eigenstate of X or Y. The family of states in which arbitrary strong squeezing ('ideal squeezing') of X or Y is possible could be called the *ideal X-Y SS*. Thus the Schrödinger IS, in particular the canonical SS [41], are ideal p-q SS. We follow the definition of X-Y SS according to [1]:

a state $|\Psi\rangle$ is X-Y SS if $\Delta X < \Delta_0$ or $\Delta Y < \Delta_0$, where Δ_0 is the lowest level at which the equality $\Delta X = \Delta Y$ can be maintained. The lowest level is reached on some eigenstate $|z_0\rangle$ of A. For the quadratures of a^k, $k = 1, 2, \ldots$, and $(a_i a_j)^k$ the lowest level is reached in the ground state $|0\rangle$. Linear and/or quadratic squeezing in ideal one-mode squared amplitude SS (eigenstates $|z; u, v\rangle$ of $ua^2 + va^2$) is considered in papers (for different ranges of parameters u, v) [17, 30, 31, 5, 43, 44]. The diagonalization of $ua^k + va^{\dagger k}$ for $k > 2$ is discussed in the very recent E-print [45].

A family of states in which the squeezing of quadratures of any product $a_i a_j$, $i, j = 1, 2, \ldots, N$, can occur should be called a family of *multimode squared amplitude SS*. An example of such a multimode SS is given by the Robertson [6] IS, which should be eigenstates of complex combinations $u_{kl;ij} a_i a_j + v_{kl;ij} a_i^\dagger a_j^\dagger$ (summation over repeated indices). These are *ideal* multimode squared amplitude SS. Multimode quadratic SS of type (b) are defined as eigenstates of all squared products $(a_i a_j)^2$. They take the form (29).

Next we consider cat-type squared amplitude SS. One example of two-mode cat-type second-order SS is considered in [20], which, however, was examined for ordinary squeezing only. Here we provide examples of multimode cat-type SS which can exhibit both quadratic and linear squeezing and other interesting statistical properties. Such SS are the following macroscopic superpositions $|\boldsymbol{\alpha}, \phi, \psi\rangle$ of the $sp(N, R)$ algebraic CS $|\boldsymbol{\alpha}, \phi\rangle$ ($|\boldsymbol{\alpha}, \phi\rangle$ are defined in equation (56)):

$$|\boldsymbol{\alpha}, \phi, \psi\rangle = \mathcal{N}(|\boldsymbol{\alpha}, \phi\rangle + e^{i\psi}|-\boldsymbol{\alpha}, \phi\rangle) \tag{61}$$

where $\mathcal{N}$ is the normalization constant, which obeys (21) and has the form

$$\mathcal{N}(\tilde{r}, \phi, \psi) = \frac{1}{\sqrt{2}}(1 + 2\tilde{\mathcal{N}}^2 e^{-\tilde{r}^2}(\cos\phi\cos(\tilde{r}^2 - \phi + \psi) + \cos(\tilde{r}^2 + \phi - \psi)))^{-\frac{1}{2}} \tag{62}$$

$\tilde{\mathcal{N}}$ being given in equation (62) and $\tilde{r} = |\boldsymbol{\alpha}| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_N|^2}$.

We demonstrate the quadrature squeezing on the example of individual mode operators a_i (linear squeezing) and a_i^2 (quadratic squeezing). Note that $|\boldsymbol{\alpha}, \phi, \psi\rangle$ are not factorized over the different modes. The variances Δp_i and Δq_i of the quadratures of the i mode annihilation operator a_i, $a_i = (q_i + ip_i)/\sqrt{2}$ are

$$\begin{aligned}
\Delta^2 p_i(\tilde{r}, r_i, \theta_i, \phi, \psi) &= \tfrac{1}{2} + \langle a_i^\dagger a_i\rangle - \mathrm{Re}\,\langle a_i^2\rangle - 2(\mathrm{Im}\,\langle a_i\rangle)^2 \\
\Delta^2 q_i(\tilde{r}, r_i, \theta_i, \phi, \psi) &= \tfrac{1}{2} + \langle a_i^\dagger a_i\rangle + \mathrm{Re}\,\langle a_i^2\rangle - 2(\mathrm{Re}\,\langle a_i\rangle)^2
\end{aligned} \tag{63}$$

where $r_i = |\alpha_i|$, $\theta_i = \arg\alpha_i$ and

$$\begin{aligned}
\langle a_i\rangle &= -2\alpha_i \mathcal{N}^2\tilde{\mathcal{N}}^2 e^{-\tilde{r}^2}\sin\phi(1 + i)(e^{-\tilde{r}^2} + \cos(\tilde{r}^2 - \phi + \psi) + \sin(\tilde{r}^2 - \phi + \psi)) \\
\langle a_i^\dagger a_i\rangle &= 4r_i^2\mathcal{N}^2\tilde{\mathcal{N}}^2(1 - \cos\phi e^{-2\tilde{r}^2} - e^{-\tilde{r}^2}(\cos\phi\sin(\tilde{r}^2 - \phi + \psi) + \sin(\tilde{r}^2 + \phi - \psi))).
\end{aligned} \tag{64}$$

As functions of θ_i the variances of p_i and q_i oscillate with period π and $\Delta p_i(\theta_i + \pi/2) = \Delta q_i(\theta_i)$. Linear squeezing is exhibited in states, for example $|\boldsymbol{\alpha}, 0, \psi\rangle$ with $r_i = 0.05$, $\tilde{r}$ close to r_i, $\theta_i = \pi/4$, $\phi = 0$ and ψ around 3.152 (see figure 1). Maximal p_i (q_i) squeezing is obtained when $\tilde{r} = r_i$ (i.e. when only one mode is excited). Here $\Delta^2 p_i \geqslant 0.275 = \Delta^2 p_i(0.05, 0.05, \pi/4, 0, 3.131) = \Delta^2 q_i(0.05, 0.05, \pi/4, 0, 3.153)$. When $\tilde{r}$ is increasing the graphics of Δp_i and Δq_i (as functions of the angles ϕ and ψ) become smoother and tend to a constant value, independent of the superposition parameters ϕ and ψ. In the above states $|\boldsymbol{\alpha}, 0, \psi\rangle$ the Mandel factor Q_i for the mode i is negative in the vicinity of $\psi = \pi$ only. By its definition the quantity $\tilde{r}^2$ coincides with the intensity of the field (the total mean number of photons $\langle \boldsymbol{a}^\dagger \boldsymbol{a}\rangle = \sum_i^N \langle a_i^\dagger a_i\rangle$) in the multimode CS $|\boldsymbol{\alpha}\rangle$. The field intensity in the multimode superposition states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ reads

$$\langle \boldsymbol{a}^\dagger \boldsymbol{a}\rangle = 4\tilde{r}^2\mathcal{N}^2\tilde{\mathcal{N}}^2(1 - \cos\phi e^{-2\tilde{r}^2} - e^{-\tilde{r}^2}(\cos\phi\sin(\tilde{r}^2 - \phi + \psi) + \sin(\tilde{r}^2 + \phi - \psi))). \tag{65}$$

We see that $\langle \boldsymbol{a}^\dagger \boldsymbol{a}\rangle$ is an increasing function of $\tilde{r}$, $\tilde{r} = |\boldsymbol{\alpha}|$.

The variances of the quadratures X_i, Y_i of the squared individual mode amplitude a_i^2, $a_i^2 = (X_i + iY_i)/\sqrt{2}$, in $|\boldsymbol{\alpha}, \phi, \psi\rangle$ is easily obtained in the form

$$\begin{aligned}\Delta^2 X_i(\tilde{r}, r_i, \theta_i, \phi, \psi) &= 1 + 2\langle a_i^\dagger a_i\rangle + \langle a_i^{\dagger 2} a_i^2\rangle + r_i^4 \cos(4\theta_i) - 2(\mathrm{Re}\,\langle a_i^2\rangle)^2 \\ \Delta^2 Y_i(\tilde{r}, r_i, \theta_i, \phi, \psi) &= 1 + 2\langle a_i^\dagger a_i\rangle + \langle a_i^{\dagger 2} a_i^2\rangle - r_i^4 \sin(4\theta_i) - 2(\mathrm{Im}\,\langle a_i^2\rangle)^2\end{aligned} \tag{66}$$

where

$$\begin{aligned}\langle a_i^2\rangle &= -4\mathrm{i}\alpha_i^2 \mathcal{N}^2 \tilde{\mathcal{N}}^2 \mathrm{e}^{-\tilde{r}^2}(\cos\phi \sin(\tilde{r}^2 - \phi + \psi) - \sin(\tilde{r}^2 + \phi - \psi)) \\ \langle a_i^{\dagger 2} a_i^2\rangle &= 2r_i^4 \mathcal{N}^2 (1 - 2\tilde{\mathcal{N}}^2 \mathrm{e}^{-\tilde{r}^2}(\cos\phi \cos(\tilde{r}^2 - \phi + \psi) + \cos(\tilde{r}^2 + \phi - \psi))).\end{aligned} \tag{67}$$

As functions of the angle θ_i the variances of X_i and Y_i oscillate with period $\pi/2$ and $\Delta X_i(\theta_i + \pi/4) = \Delta Y_i(\theta_i)$.

The variances ΔX_i and ΔY_i are squeezed if they are less than their value of 1 in the ground state $|0\rangle$. This holds, for example, in states $|\boldsymbol{\alpha}, 0, \psi\rangle$ with $\tilde{r}$ close/equal to $r_i \leqslant 1$, $\theta_i = n\pi/4$, $\phi = 0$ and ψ around zero, the minimal value of $\Delta^2 X_i$ and $\Delta^2 Y_i$ (at $\tilde{r} = r_i = 0.88$) being equal to 0.69. In the states $|0.8\mathrm{e}^{\pm \mathrm{i}\pi/4}, 0, \psi\rangle$ linear and quadratic squeezing can occur simultaneously (see graphics g_1 and g_4 on figure 2: joint X_i- and p_i- (Y_i- and q_i-) squeezing occurs in the interval $6.4 \leqslant \psi \leqslant 7.4$). In the above interval $Q_i > 0$, where Q_i is the Mandel factor for the individual mode i.

On figure 2 graphics are shown of $\Delta^2 X_i(\tilde{r}, r_i, \pi/4, 0, \psi)$ as a function of the angle ψ for fixed $r_i = 0.8$, and three different values of the total excitation parameter $\tilde{r}$, $\tilde{r} = 0.8 = r_i$ (i.e. only mode i excited, graphics g_1), $\tilde{r} = 1$ (graphics g_2) and $\tilde{r} = 1.2$ (graphics g_3). One sees, that graphics of $\Delta X_i(\psi)$ become rapidly smoother and tend to a constant value when $\tilde{r}$ is increasing.

An important statistical property of all states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ is that they are *strongly nonclassical* in the sense of the definition of [18] (discussed above), which we apply here to the total photon number (and to the conditional individual mode number) distribution in the multimode states. The total photon number distribution $p_n(\tilde{r}, \phi, \psi)$ takes the form

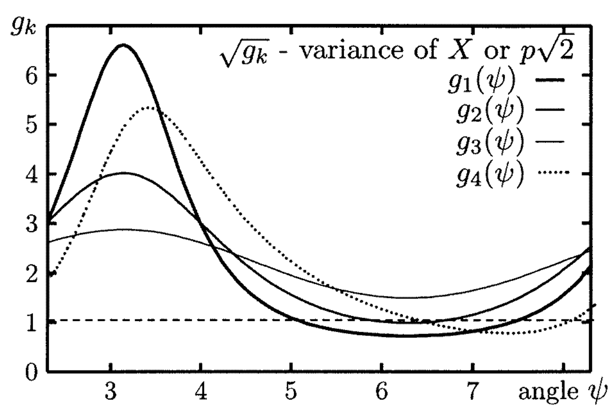

Figure 2. Squared amplitude quadrature squeezing in superpositions states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ for $r_i = 0.8$. $\Delta\tilde{X}_i = \Delta\tilde{X}_i(\tilde{r}, r_i, \theta_i, \phi, \psi)$, $\tilde{X}_i^2 = X_i$ or Y_i, $\tilde{r} = |\alpha|$. $g_1 = \Delta^2 X_i(r_i, r_i, \frac{\pi}{4}, 0, \psi) = \Delta^2 Y_i(r_i, r_i, -\frac{\pi}{4}, 0, \psi)$, $g_2 = \Delta^2 X_i(1, r_i, \frac{\pi}{4}, 0, \psi)$, $g_3 = \Delta^2 X_i(1.2, r_i, \frac{\pi}{4}, 0, \psi)$, $g_4 = 2\Delta^2 p_i(r_i, r_i, \frac{\pi}{4}, 0, \psi) = 2\Delta^2 q_i(r_i, r_i, -\frac{\pi}{4}, 0, \psi)$. Joint X and p (or Y and q) squeezing occurs in the interval $6.4 < \psi < 7.4$.

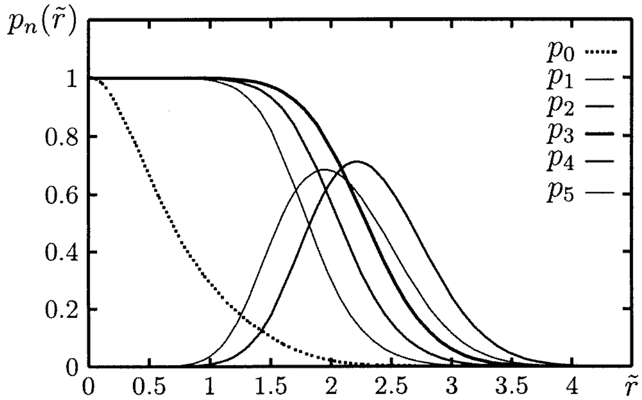

Figure 3. Probabilities $p_n(\tilde{r}, \phi, \psi)$ to find n photons in the multimode superposition states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ as functions of $\tilde{r} = |\boldsymbol{\alpha}|$ for different values of ϕ and ψ. $p_0 = p_0(\tilde{r}, \frac{\pi}{2}, \pi)$, $p_1 = p_1(\tilde{r}, \pi, -\frac{\pi}{2})$, $p_2 = p_2(\tilde{r}, 0, \pi)$,$p_3 = p_3(\tilde{r}, \pi, \frac{\pi}{2})$, $p_4 = p_4(\tilde{r}, \frac{\pi}{4}, \frac{\pi}{4})$, $p_5 = p_5(\tilde{r}, \pi, -\frac{\pi}{2})$.

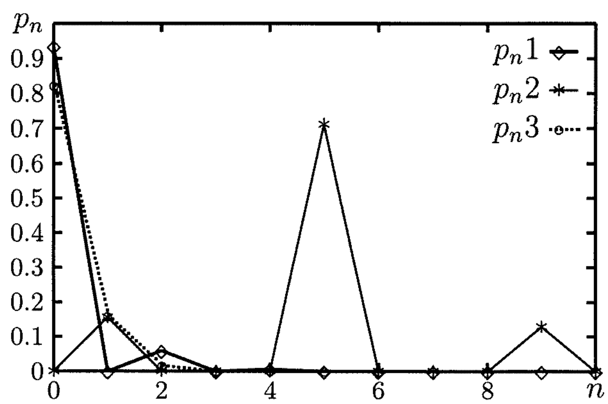

Figure 4. Oscillating photon number distributions $p_n(\tilde{r}, \phi, \psi)$ in strongly nonclassical states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ for different values of $\tilde{r}$, ϕ and ψ. $p_n1 = p_n(0.8, 0, 7.3)$ ($Q > 0$, $\Delta p \geqslant 0.38$, $\Delta X \geqslant 0.73$), $p_n2 = p_n(2.2, \pi, -\frac{\pi}{2})$ ($Q < 0$), $p_n3 = p_n(0.55, 5.0914, 0)$ ($Q = -0$, $\Delta p \geqslant 0.38$, $\Delta X > 1$).

similar to that of equation (57),

$$\begin{aligned} p_n(\tilde{r}, \phi, \psi) &= \tilde{p}_n(\tilde{r}, \phi, \psi) s_n(\phi, \psi) \\ s_n(\phi, \psi) &= |1 + (-1)^n \mathrm{e}^{\mathrm{i}\phi} + \mathrm{i}^n \mathrm{e}^{\mathrm{i}\psi} + (-\mathrm{i})^n \mathrm{e}^{\mathrm{i}(\psi - \phi)}|^2 \end{aligned} \tag{68}$$

where

$$\tilde{p}_n(\tilde{r}, \phi, \psi) = \mathcal{N}^2(\tilde{r}, \phi, \psi) \tilde{\mathcal{N}}^2(\tilde{r}, \phi) \mathrm{e}^{-\tilde{r}^2} \frac{\tilde{r}^{2n}}{n!}.$$

The factor $s_n(\phi, \psi)$ is bounded from above and as a function on n oscillates with period 4. Therefore the inequality (54) is satisfied in all states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ for those n for which s_n reaches its local maximum. This proves that all $|\boldsymbol{\alpha}, \phi, \psi\rangle$ are strongly nonclassical. Note that the factors l_{n_i} for conditional distribution of n_i ($n_{k \neq i}$ fixed) also satisfy the inequality (54).

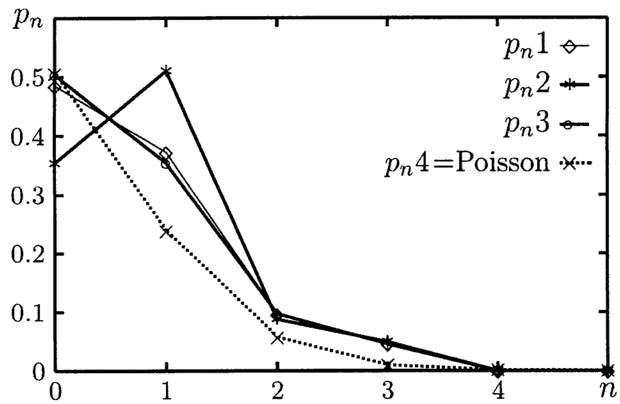

Figure 5. Nonoscillating photon number distributions in strongly nonclassical states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ for different values of $\tilde{r} = |\boldsymbol{\alpha}|$, ϕ and ψ. $p_n1 = p_n(0.55, 2.246, 0)$ ($Q < 0$, $\Delta q \geqslant 0.5$, $\Delta X \geqslant 1$), $p_n2 = p_n(0.55, 2.33, 0)$ ($Q < 0$, $\Delta q \geqslant 0.43$, $\Delta X \geqslant 1$), $p_n3 = p_n(0.55, 2.234\,384, 0)$ ($Q = +0$, $\Delta q \geqslant 0.5$, $\Delta X \geqslant 1$), $p_n4 =$ Poisson distribution with $\langle a^\dagger a\rangle = 0.685$ as in p_n3.

As in the case of $|\boldsymbol{\alpha}, \phi\rangle$ here $p_n(\tilde{r}, \phi, \psi)$ again coincides with the probability to find n photons in the one-mode states $|\tilde{\alpha}, \phi, \psi\rangle$, $|\tilde{\alpha}| = \tilde{r}$, $\tilde{\alpha} = \tilde{r}\mathrm{e}^{\mathrm{i}\tilde{\theta}}$. The distributions $p_n(\tilde{r}, \phi, \psi)$ do not depend on $\tilde{\theta}$. It can be oscillating or nonoscillating and with positive, negative or vanishing individual mode Q factor. No definite relations exist between the sign of Q, the photon number oscillations and the amplitude quadrature squeezing: all possible combinations of these three properties can be found in strongly nonclassical states $|\boldsymbol{\alpha}, \phi, \psi\rangle$. In figures 4 and 5 representative graphics of oscillating (figure 4) and nonoscillating (figure 5) photon distributions are shown. The sign of the corresponding Q and the inequalities for $\Delta q(\tilde{\theta})$ and $\Delta X(\tilde{\theta})$ for each of the graphics are also given. In the recent E-print [24] examples of classical states with oscillating photon distributions were pointed out. Thus photon number oscillations are neither necessary nor sufficient for nonclassicality of quantum states.

The Q factor is bounded, $Q \geqslant -1$, and when $Q = -1$ then the variance Δn of $\hat{n}$ is vanishing. This means [1] that the corresponding state is an eigenstate $|n\rangle$ of $\hat{n}$ (a Fock state) and $p_n(|n\rangle) = 1$. In figure 3 photon probabilities $p_n(\tilde{r}, \phi, \psi)$, $n = 0, 1, 2, 3$, are shown as functions of $\tilde{r}$ for several values of ϕ and ψ. At $\tilde{r} \to 0$ one obtains $p_n = 1$. For $N > 1$ this yields the *finite superpositions of multimode Fock states* $|\boldsymbol{n}\rangle$, $n_0 + \cdots + n_N = n$, and for the one mode case, $N = 1$,—the number state $|n\rangle$ with $n = 0, 1, 2$ or $n = 3$. We see from figure 3 that practically the states $|\boldsymbol{\alpha}, \phi, \psi\rangle$ with the corresponding ϕ, ψ coincide with Fock states $|n\rangle$, $n = 1, 2, 3$, for $|\alpha| \leqslant 0.5$ (then $p_n > 0.999\,95$). In the multimode case for $|\boldsymbol{\alpha}| \leqslant 0.5$ the specific form of superposition of several $|\boldsymbol{n}\rangle$ depends on the specific values of $|\alpha_i|$, $|\alpha_1| + \cdots + |\alpha_N| \leqslant 0.5$. If $\alpha_{k\neq i} = 0$ then all n photons/bosons are in the mode i, i.e. the Fock state is $|\boldsymbol{n}\rangle = |0, \ldots, n_i = n, 0, \ldots, 0\rangle$. There is a growing interest in obtaining Fock states from macroscopic superpositions of (so far mainly one mode) CS $|\alpha\rangle$ (see [37] and references therein). Here we provided an example, probably the first one, of obtaining Fock states of multimode systems.

6. Concluding remarks

We have constructed and discussed some properties of $sp(N, R)$ and $u(p, q)$ algebraic (algebra related) CS in the quadratic boson representation. These states are a generalization of the $su(1, 1)$ CS of BG [8] and are constructed as eigenstates of all mutually commuting Weyl lowering operators. The quadratic boson realizations of $sp(N, R)$ and $u(p, q)$ are reducible. Therefore the corresponding group related CS [10] are not overcomplete in the whole Hilbert space of states $\mathcal{H}$. The BG-type CS are very large sets and afford the possibility to resolve the unity operator in $\mathcal{H}$ by means of some subsets. We pointed out such subsets of the $sp(N, R)$ algebra related CS (and their superpositions as well) and wrote down the relations between the established $u(p, q)$ CS representations and the familiar N-mode canonical CS representation, in particular between the $su(1, 1)$ BG CS and the two-mode canonical CS representations.

The new states can exhibit interesting statistical properties, such as amplitude quadrature squeezing, sub- and super-Poissonian photon statistics and oscillations in photon number distributions. All states from the overcomplete subfamily $|\boldsymbol{\alpha}; \varphi\rangle$ of the $sp(N, R)$ BG type CS are weakly nonclassical [18] and (some of them) can exhibit amplitude quadrature squeezing as well. Strongly nonclassical [18] $sp(N, R)$ algebra related CS were also pointed out.

Noting that the BG-type CS $|\boldsymbol{z}\rangle$ cannot exhibit squeezing of the quadratures of the Weyl generators E_{ij} we anticipated that such squeezing should occur in eigenstates of E_{ij}^m, $m \geqslant 2$, which for $E_{ij}^2 = (a_i a_j)^2$ are called multimode squared amplitude Schrödinger cat states. Squared amplitude squeezing in the individual modes is demonstrated in the superpositions $|\boldsymbol{\alpha}, \phi, \psi\rangle$ of two $sp(N, R)$ CS. These are strongly nonclassical states and at small $|\boldsymbol{\alpha}|$ ($|\boldsymbol{\alpha}| < 0.5$) and for specific values of the angles ϕ, ψ practically coincide with superpositions of several multimode Fock states $|\boldsymbol{n}\rangle$ with the total number of photons/bosons $n = 1, 2$ or $n = 3$. If $\alpha_{k\neq i} = 0$ then all n photons are of the mode i, i.e. we have a single multimode Fock state. The Fock state engineering via discrete superpositions of canonical CS $|\alpha\rangle$ is of current interest in the literature (one mode mainly). We have shown that discrete superpositions of multimode canonical CS are naturally encompassed in the framework of $sp(n, R)$ BG-type CS and their linear combinations. The weakly nonclassical $sp(N, R)$ BG CS $|\boldsymbol{\alpha}; \varphi\rangle$ can be generated from CS $|\boldsymbol{\alpha}\rangle$ by means of the second kind of (unitary) squeeze operator. This should be considered in greater detail elsewhere.

Acknowledgment

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Appendix

A.1. The correspondence rule between the N-mode canonical CS and $u(p, q)$ BG-type CS representations

The multimode CS $|\boldsymbol{\alpha}\rangle$ are overcomplete in the N-mode boson system Hilbert space $\mathcal{H}$, spanned by the number states $|\boldsymbol{n}\rangle = |n_1, \ldots, n_N\rangle$. In the canonical CS representation a state $|\Psi\rangle$ is represented by the analytic function $F_{\mathrm{CCS}}(\boldsymbol{\alpha}; \Psi)$ of N variables α_i, $i = 1, \ldots, N$,

$$F_{\mathrm{CCS}}(\boldsymbol{\alpha}, \Psi) = \langle \boldsymbol{\alpha}^* \| \Psi\rangle \qquad \|\boldsymbol{\alpha}\rangle = \sum_{n_1,\ldots,n_N} \frac{\alpha_1^{n_1} \ldots \alpha_N^{n_N}}{\sqrt{n_1! \ldots n_N!}} |\boldsymbol{n}\rangle. \tag{69}$$

The $u(p,q)$ BG-type CS $\|\boldsymbol{z}; l, p, q\rangle$, equation (41), are overcomplete in subspaces $\mathcal{H}_l$, satisfying the resolution unity equation (42). Hereafter if a state $|\Psi\rangle \in \mathcal{H}_l \subset \mathcal{H}$, then in the $u(p,q)$ CS representation this state is represented by the analytic functions of $N-1$ variables z_k, $k = 1, \ldots, N-1$,

$$F_l(p,q,\boldsymbol{z};\Psi) = \langle q, p, l; \boldsymbol{z}^*\|\Psi\rangle. \tag{70}$$

The relation between the two representatives of $|\Psi\rangle$ immediately follows from the expansion (37) and equations (40) and (44):

$$F_{\mathrm{CCS}}(\boldsymbol{\alpha},\Psi) = \sum_{l=-\infty}^{\infty} \alpha_N^{-l} F_l(p,q,\boldsymbol{z};\Psi) \qquad z_k \text{ given by equation (40).} \tag{71}$$

This formula is efficient for the transition from $\{F_l\}$ to F_{CCS} if one knows the representatives $F_l(p,q,\boldsymbol{z};\Psi)$. In the opposite direction the transition formula is easily obtained from (70), (37) and the orthogonality between $\mathcal{H}_l$ and $\mathcal{H}_{l'\neq l}$,

$$F_l(p,q,\boldsymbol{z}';\Psi) = \frac{1}{\pi^N}\int \mathrm{d}^2\boldsymbol{\alpha}\,\alpha_N^l\langle q,p,l;\boldsymbol{z}'^*\|\boldsymbol{z}(\boldsymbol{\alpha}); l,p,q\rangle \mathrm{e}^{-|\boldsymbol{\alpha}|^2} F_{\mathrm{CCS}}(\alpha;\Psi) \tag{72}$$

where $\boldsymbol{z}(\boldsymbol{\alpha})$ is given according to (40) and $\|\boldsymbol{z}; l, p, q\rangle$ is the state (41).

The $u(p,q)$ CS representation is not yet fully specified (this could be a subject for a separate work), except for the case of $p = 1 = q$ ($N = 2$ when it coincides with the well known $su(1,1)$ BG CS representation [8]. In this case the relation (71) is rewritten in the simpler form

$$F_{\mathrm{CCS}}(\alpha_1,\alpha_2,\Psi) = F_{l=0}(z;\Psi) + \sum_{l=1}^{\infty}(\alpha_1^l + \alpha_2^l)F_l(z;\Psi) \qquad z = \alpha_1\alpha_2. \tag{73}$$

The BG representation is given [8] in terms of Bargman index k, not in terms of l: $|\Psi\rangle \to F_{\mathrm{BG}}(z,k;\Psi)$. The relation between l and k is $l = \pm\sqrt{4k(k-1)+1}$, or

$$k = \tfrac{1}{2}(1+|l|) \qquad l = n_1 - n_2. \tag{74}$$

One has $F_{l\leqslant 0}(z;\Psi) = F_{\mathrm{BG}}(z, k = (1+|l|)/2;\Psi)$, $F_{l>0}(z;\Psi) = F_{\mathrm{BG}}(z, k = (1+l)/2;\Psi)$ and

$$F_{\mathrm{CCS}}(\alpha_1,\alpha_2;\Psi) = F_{\mathrm{BG}}(z, k = \tfrac{1}{2};\Psi) + \sum_{k\geqslant 1}(\alpha_1^{2k-1} + \alpha_2^{2k-1})F_{\mathrm{BG}}(z,k;\Psi) \qquad z = \alpha_1\alpha_2. \tag{75}$$

The relation (74) stems from the definition of k by means of the Casimir operator: $C_2 = K_3^2 - \frac{1}{2}(K_+K_- + K_-K_+) = k(k-1)$. In the two-mode $su(1,1)$ representation (9) we have $C_2 = -\frac{1}{4} + L^2/4$ which tell us that both l and $-l$ lead to the same values of C_2, that is the representations realized in the subspaces with $\pm l$ are equivalent. However, for the transitions between canonical CS and BG representations the sign of l is significant and is taken into account in (73) by the identification of the standard $su(1,1)$ notation $|n+k,k\rangle$ of the eigenstates of K_3 once with the two-mode Fock state $|n+|l|,n\rangle$ (the first term in the sum in (75)) and second with $|n,n+|l|\rangle$ (the second term in the sum in (75)). Keeping in mind the latter identification rule we can express the two-mode canonical CS $|\alpha_1,\alpha_2\rangle$ in terms of BG CS $|z;k\rangle$,

$$|\alpha_1,\alpha_2\rangle = |z;k=\tfrac{1}{2}\rangle + \sum_{k\geqslant 1}(\alpha_1^{2k-1} + \alpha_2^{2k-1})|z;k\rangle \qquad z = \alpha_1\alpha_2. \tag{76}$$

A.2. On the overcompleteness of eigenstates of A^{2^n}

Let $\{|z\rangle\}$ be an overcomplete family of eigenstates of an N non-Hermitian operator A_i, $A_i|z\rangle = z_i|z\rangle$,

$$1 = \int d\mu\,(z, z^*)|z\rangle\langle z| \tag{77}$$

where $d\mu(z, z^*) = F(|z_1|, \ldots, |z_N|)d^2z$. We note that the requirement of the weight function F not to depends on the phases of z_i. This, for example, holds for the N-mode canonical CS, $sp(N,R)$ and $u(p,q)$ CS (23) and (41). Consider the sequence of families $|z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle$,

$$|z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle = \cos\varphi_n|z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle + \mathrm{i}\sin\varphi_n|-z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle \tag{78}$$

where φ_k, $k = 1, \ldots, n$, are angle parameters, n is any positive integer and z^{2^n} is the N-component column $(z_1^{2^n}, \ldots, z_N^{2^n})$ of eigenvalues of powers $A_i^{2^n}$ of operators A_i,

$$A_i^{2^n}|\pm z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle = \pm z_i^{2^n}|\pm z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle. \tag{79}$$

$|z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle$ are superpositions of 2^n states $|z\rangle$ with z_i on circles of radius $|z_i|$. If all φ_k are integer multiple of $\pi/2$ then one obtains the states $|\pm z\rangle$. Independent parameters are $z, \varphi_1, \ldots, \varphi_n$, therefore one could also use the notation $|z; \varphi^{(n)}\rangle$ (as in section 3).

Theorem. If in equation (77) $d\mu(z, z^*) = F(|z_1|, \ldots, |z_N|)\,d^2z$ then

$$1 = \int d\mu\,(z, z^*)|z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle\langle\varphi_1, \ldots, \varphi_n; z^{2^n}| \qquad n = 0, 1, 2, \ldots. \tag{80}$$

Proof. The theorem is valid for $n = 0$ by construction. It is not difficult to check directly, that it is valid for several $n > 0$. Now suppose that it is valid for $n-1$. Then we shall prove that it is also valid for n. Indeed, using the definition (78) and noting that $-z_j^{2^n} = iz_j^{2^{n-1}}$ we obtain for the projectors in (80) the expression,

$$\begin{aligned}
|z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle\langle\varphi_1, \ldots, \varphi_n; z^{2^n}| &= \cos^2\varphi_n|z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle\langle\varphi_1 \ldots, \varphi_{n-1}; z^{2^{n-1}}| \\
&\quad + \sin^2\varphi_n|-z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle\langle\varphi_1 \ldots, \varphi_{n-1}; -z^{2^{n-1}}| \\
&\quad + \mathrm{i}\cos\phi_n\sin\varphi_n|z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle\langle\varphi_1, \ldots, \varphi_{n-1}; -z^{2^{n-1}}| \\
&\quad - \mathrm{i}\cos\phi_n\sin\varphi_n|-z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle\langle\varphi_1, \ldots, \varphi_{n-1}; z^{2^{n-1}}|.
\end{aligned} \tag{81}$$

We substitute this expression into equation (77) and then in the second and in the last integral change the integration variables z_i to $z_j\exp[i\pi/2^{n-1}]$ (rotation on angle $\pi/2^{n-1}$). Then we note that under such rotation the eigenvalues $z_j^{2^{n-1}}$ of $A^{2^{n-1}}$ change the sign, i.e. $|z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle \to |-z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle$. This yields the cancellation of the last two integrals and the coincidence of the first two ones in view of the rotational invariance of the resolution unity measure $d\mu(z, z^*) = F(|z_1|, \ldots, |z_N|)d^2z$. We obtain

$$\begin{aligned}
&\int d\mu\,(z, z^*)|z^{2^n}; \varphi_1, \ldots, \varphi_n\rangle\langle\varphi_1, \ldots, \varphi_n; z^{2^n}| \\
&\qquad = \int d\mu\,(z, z^*)|z^{2^{n-1}}; \varphi_1, \ldots, \varphi_{n-1}\rangle\langle\varphi_1, \ldots, \varphi_{n-1}; z^{2^{n-1}}| = 1.
\end{aligned} \tag{82}$$

□

A.3. On the uniqueness of the resolution unity measures $\mathrm{d}\mu(\boldsymbol{z}, l, p, q)$ *for* $u(p, q)$ *CS*

The resolution unity measure for a given continuous family of states is generally not unique. It could be unique if a certain constraint is imposed on the class of admissible measures. For example, the requirement of *invariance* of the measure on the group manifold under the group action determines it uniquely [11]. As a result the resolution unity measure for the group related CS is unique if it is invariant under the group action. For canonical SS families of noninvariant resolution unity measures have been constructed in [46]. Canonical SS minimize the Schrödinger uncertainty relation and can be regarded as group related CS for the semidirect product of $SU(1, 1)$ and the Heisenberg–Weyl group [46].

In this section we establish that the resolution unity measure for the $u(p, q)$ CS (41) is uniquely determined by the requirement of the weight function $F(z_1, \ldots, z_{N-1})$ to be a smooth function of $|z_i|$ and independent of $\arg z_i$, $i = 1, 2, \ldots, N-1$; such is our weight function in equation (42).

Suppose that there exists another function $F'(|z_1|, \ldots, |z_{N-1}|; l, p, q)$ such that the new measure $\mathrm{d}\mu' = F'\mathrm{d}^2\boldsymbol{z}$ resolves the unity 1_l as in equation (42). Then we should have

$$0 = \int \mathrm{d}^2\boldsymbol{z}\,[F(|\tilde{\boldsymbol{z}}_p|, |\tilde{\boldsymbol{z}}_q|; l, p, q) - F'(|z_1|, \ldots, |z_{N-1}|; l, p, q)]\|\boldsymbol{z}; l, p, q\rangle\langle q, p, l; \boldsymbol{z}\|. \tag{83}$$

Substituting the expansion (41) of $\|\boldsymbol{z}; l, p, q\rangle$ and integrating with respect to angles $\varphi_i = \arg z_i$ we obtain that the difference function

$$\Phi(r_1, r_2, \ldots, r_{N-1}) \equiv F(\tilde{r}_p, \tilde{r}_q; l, p, q) - F'(|z_1|, \ldots, |z_{N-1}|; l, p, q)$$

where $\tilde{r}_p \equiv |\tilde{\boldsymbol{z}}_p| = \sqrt{r_1^2 + \cdots + r_p^2}$ and $\tilde{r}_q \equiv |\tilde{\boldsymbol{z}}_q| = \sqrt{r_{p+1}^2 + \cdots + r_{N-1}^2}$, should be orthogonal to the monomials

$$r_1^{2n_1+1} \ldots r_{N-1}^{2n_{N-1}+1} \qquad r_i = |z_i| \qquad i = 1, \ldots, N-1 \qquad n_i = 1, 2, \ldots.$$

Changing the integration variables and redenoting r_i^2 again as r_i one can write this orthogonality in the form

$$\int_0^\infty \mathrm{d}r_1 \ldots \mathrm{d}r_{N-1}\Phi(r_1, \ldots, r_{N-1}) r_1^{n_1} \ldots r_{N-1}^{n_{N-1}} = 0 \tag{84}$$

where $n_i = 1, 2, \ldots$, $i = 1, \ldots, N-1$. Equation (84) implies that $\Phi(r_1, \ldots, r_{N-1})$ is decreasing exponentially as the total radius $r_1^2 + \cdots + r_{N-1}^2$ tends to ∞. This means that the integral $\int_0^\infty \Phi^2\, \mathrm{d}r_1 \ldots \mathrm{d}r_{N-1}$ is finite. It also follows from equation (84) that Φ is orthogonal to any function $f(r_1 \ldots r_{N-1})$ which admits power expansion in terms of r_i, $i = 1, 2, \ldots N-1$,

$$\int_0^\infty \mathrm{d}r_1 \ldots \mathrm{d}r_{N-1}\Phi(r_1, \ldots, r_{N-1}) f(r_1 \ldots r_{N-1}) = 0. \tag{85}$$

This implies that $\Phi \equiv F - F' = 0$ almost everywhere. Indeed, if $\Phi \neq 0$ it must be nonpositive definite (in order to obey (84)) and if Φ is well behaved (it is sufficient to be continuous) we could find f which is negative in the domains where $\Phi < 0$. However, then we could not maintain (85), unless $F = F'$ almost everywhere. We suppose in (85) that the integral of the power series of f is a sum of terms of the type of (84). This is ensured if Φ is a smooth function (i.e. all derivatives finite) of $r_1, \ldots, r_{N-1}$ (our F, equation (43), is such a function). In this case we can take in (85) $f = \Phi$. Then we obtain that F and F' should coincide pointwise. Thus the resolution unity measure (43) is unique within the set of smooth functions of $|z_1|, \ldots, |z_{N-1}|$.

A.4. Proof of the representation (46) of the Bessel function $K_\nu(z)$

In the case of $q = 1$ ($p = N - 1$) and $-l \geqslant 0$ our measure function F, equation (43), depends on $r_1, \ldots, r_p$ through $|\boldsymbol{z}| = [|z_1|^2 + \cdots + |z_p|^2]^{1/2} \equiv \tilde{r}_p$ and it is a smooth and positive function of $r_1, \ldots, r_p$. The measure function of [7] $F' \sim \tilde{r}_p^{-l-p+1} K_{-l-p+1}(2\tilde{r}_p)$ is also smooth and positive [26]. Therefore the difference Φ of these two functions is smooth and in view of (84) and the result of the preceding section they have to coincide pointwise. This proves formula (46) for $\mathrm{Im}\, z = 0, \mathrm{Re}\, z > 0$ and $\nu = -l - p + 1 = 0, \pm 1, \ldots$.

Let us consider the right-hand side of (46) as a definition of a new function $F(z;\nu)$, z complex, ν real. The integral is convergent for $\mathrm{Re}\, z > 0$ and the function $F(z;\nu)$ is evidently analytic with respect to z and ν. The Bessel function $K_\nu(z)$ is analytic and regular everywhere except of the negative half of the real line in z-plain [26]. We proved in the above that the two analytic functions $F(z;\nu)$ and $K_\nu(2z)$ ($\nu = 0, \pm 1, \ldots$) coincide on the positive part of the real line of z. Then they coincide in the whole domain of analyticity in z-plain. Numerical computations show that formula (46) holds for complex ν as well. In conclusion let us note that the integral in the right-hand side of equation (43) correctly defines (under replacements $|\tilde{\boldsymbol{z}}_p| \to z_1$, $|\tilde{\boldsymbol{z}}_q| \to z_2$) analytic functions $F(z_1, z_2; l, p, q)$ of the two variables z_1 and z_2, $\mathrm{Re}\, z_{1,2} > 0$. At $z_2 = 0$, $q = 1$ we have $F(z_1, 0; l, p, 1) = 2\pi^{-p}|z_1|^{1-l-p} K_{1-l-p}(2z_1)$.

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